Notes

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This version: 13th November 2020

These are notes that I am writing along my summer project. They cover the material needed to compute primordial inflationary correlators. Three and four points correlators for $P(X, \phi)$ inflationary Lagrangians are computed. The calculations needed to obtain the results of Maldacena's paper on primordial non-gaussianity ([20]) are reproduced. It also summarises the spinor-helicity formalism for scattering amplitudes.

These notes should be coherent without any knowledge of courses of Cambridge's Part III of the Mathematical Tripos, but they do require understanding of Part II courses. Particularly important are Principles of Quantum Mechanics and Cosmology; to a lesser extent, Classical Dynamics and General Relativity.

These notes are far from original, and are heavily based on notes for relevant Part III courses. Credit will be added where appropiate. (But all errors are almost surely mine.)

The calculations of correlators at the end are beyond Part III content (I think), but no originality is claimed.

I am grateful to Dr Enrico Pajer (DAMTP) for taking me as a summer student, and to Harry Goodhew for answering my (sometimes tricky) questions. This project was partially funded by Summer Research in CMS program of the Faculty of Mathematics.

An updated version of these notes and other related documents can be found at:
https://mariaalegriagutierrez.wordpress.com/summer-research-2020

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1 Background Cosmology

The first goal of this section is to review the FLRW spacetime, the Friedman, continuity and acceleration equations de Sitter spacetime. Secondly, we give a more formal treatment than in part II to the action for a canonical scalar field, the slow-roll conditions and slow-roll solution for inflation. I follow closely the Part III Cosmology notes [5].

Conventions for this section: "Natural" units (in which $\hbar = 1 = c$) are used throughout. The reduced Planck mass $M_{Pl} = (8\pi G)^{-1/2}$ will be kept explicit. I use the mostly plus signature (-,+,+,+).

1.1 General Relativity

Here we recall some important results from GR.
To start with, two principles from which GR can be derived:

- The *equivalence principle*: Free falling observers do not feel the effect of gravitation. In an open set around any spacetime point we can choose the *locally inertial frame*(LIF), ie (*normal*) coordinates at which the metric tensor is approximately Minkowski: $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_\gamma g_{\mu\nu} = 0$.

- The *covariance principle*: Equations must be invariant in form under a change of coordinates. The laws are obtained from those of special relativity by promoting the Minkowski $\eta_{\mu\nu}$ metric to the spacetime metric $g_{\mu\nu}$ and partial derivative ∂_μ to covariant derivatives ∇_μ.

A common strategy for GR is:

1. Write down the equations governing a (sufficiently small) system in the abscense of gravity.

2. Re-write them in a covariant way.

The Geodesic Equation: A curve $x^\mu(u)$ parametrised by an affine parameter u is a *geodesic* iff it obeys

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{du}\frac{dx^\beta}{du} = 0 \tag{1.1}$$

Here $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta})$ are the Christoffel symbols for the Levi-Civita connection.
The **Riemann** tensor is

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

The **Ricci tensor** is $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$ while the **Ricci Scala**r is $R = R^{\mu}_{\mu}$.

The contracted Bianchi identity for the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

is

$$\boxed{\nabla^\mu G_{\mu\nu} = 0} \tag{1.2}$$

The Einstein Equations and the Energy-Momentum tensor:
Firstly, we'll recall what we know about the Energy-Momentum tensor.

- In vacuum, $T_{\mu\nu} = 0$
- In the comoving LIF, the energy-momentum tensor for a perfect fluid is diagonal and isotropic: T^{μ}_{ν} =diag(-ρ,p,p,p). Its covariant form is

$$\boxed{T^{\mu\nu} = (\rho + p)u^\mu u^\nu + g^{\mu\nu}p} \tag{1.3}$$

where u^μ is the 4-velocity, satisfying $u_\mu u^\mu = -1$.
This relationship can be inverted to find ρ, p and u^μ in terms of $T_{\mu\nu}$

$$\rho \equiv \frac{1}{4}\left(\sqrt{12T_{\mu\nu}T^{\mu\nu} - 3T^2} - T\right) \tag{1.4}$$

$$p \equiv \frac{1}{12}\left(\sqrt{12T_{\mu\nu}T^{\mu\nu} - 3T^2} + 3T\right) \tag{1.5}$$

$$u_\mu u_\nu \equiv \frac{T_{\mu\nu} - g_{\mu\nu}p}{\rho + p} \tag{1.6}$$

where $T \equiv T^{\mu}_{\mu}$. [To prove this, it's easier to work in the LIF. Since the above expressions are covariant, it suffices to show that they're correct in the LIF.]

In GR the metric is dynamical and its evolution is dictated by the *Einstein Equations*:

$$\boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu} = M_{Pl}^{-2} T_{\mu\nu}} \tag{1.7}$$

Here we have not added a cosmological constant term (or rather, we have included it in $T_{\mu\nu}$).
Let's recall what we did last year:

- We took conservation of $T^{\mu\nu}$, i.e

$$\nabla_\mu T^{\mu\nu} = 0 \tag{1.8}$$

  as a postulate of GR.

- We also took the EE's (1.7) as a postulate of GR, with some intuition given by *Lovelock's theorem* [3].

- Conservation of $T^{\mu\nu}$ is implied by the EE's (1.7), and the contracted Bianchi identities (1.2).

Now we get into new material. We'll use an action S for GR to derive the EE's.

To avoid confusion... If you recall GR from last year, we <u>did</u> use an action. We derived the geodesic equations by extremizing the following action:

$$S = \int L d\lambda \tag{1.9}$$

Here $L = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$ was our Lagrangian and λ an affine parameter (usually proper time for timelike curves or proper length for spacelike curves).
There are few things worth noting here:

- This is a one-dimensional integral (over one parameter: λ).

- We only considered variations of this action with respect to the curve $x^\mu(\lambda)$.

- Setting such variation $\delta S/(\delta x^\mu)$ to zero gave us the geodesic equations. These tell us about the geodesic curves in such spacetime, <u>not</u> about the metric tensor $g_{\mu\nu}$.

Obtaining the EE's from an action. Finally coming to the new stuff. I'll follow Tong's notes for Part III GR [8] very closely.
All fundamental theories of physics are described by action principles. Gravity is no different.

- We would like to have an action to give us the dynamics of the metric.
- If you did Electrodynamics in Part II [4], what we want is analogous to the action S_{EM} for the electromagnetic fields. It was an integral over spacetime of a Lorentz invariant *Lagrangian* density. When extremizing the action with respect to the 4-vector potential A_μ, we obtained Maxwell's equation. Maxwell's equations tell us the dynamics of the EM field.

The *Einstein-Hilbert action* is

$$S = \frac{1}{2} M_{Pl}^2 \int d^4x \sqrt{-g} R \tag{1.10}$$

Here g is the determinant of the metric tensor $g_{\mu\nu}$. The factor of $\sqrt{-g}$ gives an appropriate volume form (this will be more obvious when we restrict to the FLRW metric, for which $\sqrt{-g} = a^3$).
We now vary this action. We shift

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \delta g_{\mu\nu}(x) \tag{1.11}$$

Writing $R = g^{\mu\nu} R_{\mu\nu}$, we obtain

$$\delta S = \frac{1}{2} M_{Pl}^2 \int d^4x \left((\delta\sqrt{-g}) g^{\mu\nu} R_{\mu\nu} + \sqrt{-g} R_{\mu\nu} (\delta g^{\mu\nu}) + \sqrt{-g} (\delta R_{\mu\nu}) g^{\mu\nu} \right) \tag{1.12}$$

We now use to results that the reader is invited to prove (or consult in Tong's notes [8]): The variation of the volume element is given by

$$\delta\sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \tag{1.13}$$

while the variation of the Ricci tensor $R_{\mu\nu}$ is a total derivative:

$$g^{\mu\nu} \delta R_{\rho\nu} = \nabla_\mu \delta\Gamma^\mu_{\nu\rho} - \nabla_\nu \delta\Gamma^\mu_{\rho\mu} \equiv \nabla_\mu X^\mu \tag{1.14}$$

Hence the variation of the action (1.12) becomes:

$$\delta S = \frac{1}{2}M_{Pl}^2 \int d^4x\sqrt{-g}\left[\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} + \nabla_\mu X^\mu\right] \tag{1.15}$$

We can use the divergence theorem. It could be reasonable to ignore the boundary term, and we will do so. Hence:

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0 \implies 0 = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \equiv G_{\mu\nu} \tag{1.16}$$

These are the Einstein Equations (1.7) for vacuum (and without a cosmological constant).
If our theory has a cosmological constant λ, then the corresponding action is:

$$S = \frac{1}{2}M_{Pl}^2 \int d^4x\sqrt{-g}(R - 2\Lambda) \tag{1.17}$$

We want to understand how fields behave on spacetime. If the matter theory is described by an action S_M (which depends on both the matter fields and the metric), we need to consider the combined action:

$$S = \frac{1}{2}M_{Pl}^2 \int d^4x\sqrt{-g}(R - 2\Lambda) + S_M \tag{1.18}$$

The energy momentum is defined by

$$T^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} \tag{1.19}$$

The sign depends on conventions. I use an opposite sign to the part III Cosmology notes [5]. This choice agrees with the Part III GR notes [8]. It also ensures that the EE's have the same form as in Part II GR [2], namely (1.7). This definition of $T_{\mu\nu}$ will give us the Einstein Equations (1.7) with a cosmological constant. Consider a variation of the metric, as in (1.11). Then, using (1.13) and (1.15) we get

$$\delta S = \frac{1}{2}M_{Pl}^2 \int d^4x\sqrt{-g}(G_{\mu\nu} + \Lambda g_{\mu\nu})\delta g^{\mu\nu} - \frac{1}{2}\int d^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu} \tag{1.20}$$

and hence setting $\delta S = 0$ gives us the Einstein Equations with a cosmological constant.

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = M_{Pl}^2 T_{\mu\nu}} \tag{1.21}$$

1.2 A Homogeneous and Isotropic Expanding Universe

This section recalls the some important concepts from cosmology, [1]. We also saw most of them (if not all) in the relevant chapter of GR [2].

1.2.1 Symmetric Spaces

Refer to the notes [5], or to [15] for a longer treatment. This won't be very important for our purpose -these are things from GR, eg Killing vectors-, but the metrics we use to describe the expansion of the universe are constrained by this.

1.2.2 The Friedmann-Lemaitre-Robertson-Walker metric

Recall from Part II the *Cosmological Principle* [1], [2]:
At a given moment in time, the universe is spatially *homogeneous* and *isotropic* when viewed on a large scale.
This assumption is supported evidence from the CMB, redshift surveys, etc.
In Part II GR, we derived a metric for the universe on large scales, from the assumption of *maximally symmetric spacetime*. After some rescaling, we obtained the FLRW metric:

$$ds^2 = -dt^2 + a^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega_2^2\right] \tag{1.22}$$

where the *spatial curvature* K is either -1 (open hyperbolic space), +1 (closed space or sphere), or 0 (flat space). (To get this expression from the form we used in Cosmology, we need to rescale r and a.)
Here $d\Omega_2^2 = \sin^2\theta d\phi^2 + d\theta^2$.
In (t, r, θ, ϕ) coordinates, the metric tensor is given by

$$g_{\mu\nu} = \text{diag}\left(-1, \frac{a^2}{1 - Kr^2}, a^2 r^2, a^2 r^2 \sin^2\theta\right)$$

.

The expansion of the universe is determined by $a(t)$, the *scale factor*. The *Hubble parameter* is $H \equiv \dot{a}/a$ (note that it is a function of time).

There is no evidence of spatial curvature in our universe and current upper bounds constrain it to be at a sub-percent level. Hence, unless otherwise specified, I will assume $K = 0$. In this case, the metric tensor in cartesian coordinates (t, x, y, z) reduces to

$$g_{\mu\nu} = \text{diag}(-1, a^2, a^2, a^2) \tag{1.23}$$

1.2.3 Dynamical Equations

Continuity Equation
The Einstein Equations (1.7) imply the covariant conservation of energy and momentum current, i.e. $\nabla_\mu T^{\mu\nu} = 0$. The spatial components are trivial because of isotropy. The time component gives the continuity equation:

$$\boxed{\dot{\rho} + 3H(\rho + p) = 0} \tag{1.24}$$

In cosmology, the equation of state $p = p(\rho)$ is usually assumed to be of the form

$$p = \omega\rho \tag{1.25}$$

This leads to $\rho \propto a^{-3(1+\omega)}$ for a single fluid component.
Friedmann equation
The Riemman and Ricci tensors for the FLRW metric can be computed from their definition. The 00 component of the EE's (1.7) gives the *Friedmann Equation*:

$$\boxed{3M_{Pl}^2(H^2 + K/a^2) = \rho = \sum_\omega \rho_\omega} \tag{1.26}$$

where the sum runs over all the constituents of the universe.

- This is the same form we quoted in Cosmology.
- This is also as in Part II GR, except that we are not explicitly writing the term corresponding to a cosmological constant.We included it ρ, recalling that dark energy corresponds to a fluid with $\omega = -1$.
- For a universe without curvature $K = 0$ and a single fluid component, the solution is

$$a = \left[\frac{3}{2}(1+\omega)H_0 t\right]^{\frac{2}{3(1+\omega)}} \implies H(t) = \frac{2}{3(1+\omega)t} \tag{1.27}$$

Dividing both sides of the Friedmann equation (1.26) by the critical density $\rho_c(t) \equiv 3M_{Pl}^2 H^2$ we obtain

$$1 - \Omega_k = \sum_a \Omega_a \tag{1.28}$$

where $\Omega_k \equiv -K/(H^2 a^2)$ and $\Omega_a \equiv \rho_a/\rho_c$. (Recall from Cosmology that the curvature term can be thought of as a fluid with $\omega = -1/3$.)
Multiplying the Friedmann equation (1.26) by a^2, differentiating the resulting expression and using the continuity equation (1.24) gives us the *acceleration equation*:

$$\boxed{M_{Pl}^2 \frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p)} \tag{1.29}$$

from which we can get as well the evolution for the Hubble parameter:

$$-\dot{H} M_{Pl}^2 = \frac{1}{2}(\rho + p) \tag{1.30}$$

The Null Energy Condition (NEC): A certain form of matter with energy-momentum tensor $T_{\mu\nu}$ satisfies the NEC if for every null vector $N^\mu N_\mu = 0$ one has $T_{\mu\nu} N^\mu N^\nu \geq 0$. Violations of the NEC are often associated with pathologies. Using the perfect fluid parameterization (1.3), this implies $\rho + p \geq 0$. For a single-fluid component, the NEC becomes

$$\omega \geq -1 \tag{1.31}$$

Informally, if your theory violates the NEC, then you most likely run into trouble because your theory will have instabilities. There are details to this and it is not a sure thing. But we usually impose the NEC to avoid issues.

1.3 Motivations for Inflation

Now we move into inflation. (We didn't study it in GR, but we did in Cosmology [1].)

1.3.1 Curvature Problem

One of the old background problems.
We do not observe any spatial curvature in our universe, despite the fact that

curvature dilutes more slowly (as a^{-2}) than radiation (as a^{-4}) and matter (as a^{-3}). In fact, we saw that in part II that a flat universe is *unstable*. See the notes for Part III for further discussion, bounds, etc

1.3.2 Horizon Problem

A second background problem of the Hot Big Bang model is that the homogeneity of the observed universe on large scales is at odds with the decelerated expansion history. Cosmological observations of far away objects allow us to see homogeneity in regions in the past that are much larger than the particle horizon at any time. This violates causality.

1.3.3 New perturbation problems

There are "new" problems with the hot Big Bang models, which they were not known 40 years ago (the data was not good enough). They are also "new" because we didn't cover them last year. The following is based mainly on [5]

Phase coherence problem: By observing distances at $z \gg 1$, we see scales much larger than the Hubble radius at the corresponding time. There are perturbations in our universe at these superHubble scales (wavelength $\lambda > 1/H$). Remarkably, these oscillate in exact synchronicity: they have all the same phase. This is the *phase coherence* of cosmological perturbations. This is problematic because on such super-horizon scales no causal mechanism can be devised to "synchronize" the phases. Their coherence would be a very unlikely coincidence.

The Monopole Problem [6], [7]: Most Grand Unified Theories (GUT)[1] predict the production of "relic particles". These theories predict a number of heavy, stable particles that have not been observed in nature, such as magnetic monopoles. Monopoles should have persisted to the present day but all searches for them have failed.
A period of inflation that occurs below the temperature where magnetic monopoles can be produced would offer a possible resolution of this problem: monopoles would be separated from each other as the Universe around

[1] They propose that at high temperatures (such as in the early universe) the electromagnetic force, strong, and weak nuclear forces are merged into a single force.

them expands, potentially lowering their observed density by many orders of magnitude.

Scale Invariance Problem: There is another problem with the perturbed universe: the amplitude of perturbations observed is approximately the same on all cosmological scales. This surprising feature of the *primordial perturbations* is known as *scale invariance*. The definition of this is that a field ϕ obeys scale invariance if for every $\lambda \in \mathbb{R}$ and every $n \in \mathbb{N}$, we have:

$$\langle \phi(\vec{x}_1) \dots \phi(\vec{x}_n) \rangle = \langle \phi(\lambda \vec{x}_1) \dots \phi(\lambda \vec{x}_n) \rangle$$

We would like to see scale invariance emerging from a scaling symmetry of the primordial physics that generated the perturbations.
So we not only we need an accelerating space in the very early universe, but we also need it to give rise to scale invariance.
A very simple and elegant solution is to assume that, during some primordial era, the background spacetime was well approximated by *de Sitter space* in flat space.

1.3.4 de Sitter Spacetime

A cosmological constant $\Lambda > 0$ supports a de Sitter (dS) solution:

$$a(t) \propto \cosh \sqrt{\frac{\Lambda}{3}} t \qquad \text{(for } K = +1) \tag{1.32}$$

$$a(t) \propto \exp\left\{ \sqrt{\frac{\Lambda}{3}} t \right\} \qquad \text{(for } K = 0) \tag{1.33}$$

$$a(t) \propto \sinh \sqrt{\frac{\Lambda}{3}} t \qquad \text{(for } K = -1) \tag{1.34}$$

We are mainly interested in the case $K = 0$.

In such case, case we have:

$$\boxed{ds^2 = \frac{-d\tau^2 + dx^i dx^j \delta_{ij}}{\tau^2 H^2} = -dt^2 + e^{2Ht} dx^i dx^j \delta_{ij}} \tag{1.35}$$

where we have used conformal time τ given by $d\tau = dt/a$. Here we have $a = e^{Ht} (= -1/(H\tau))$ with constant Hubble parameter $H \equiv \sqrt{\frac{\Lambda}{3}}$.

One of the ten isometries of de Sitter space (which is maximally symmetric) is the *dilatation* symmetry:

$$\tau \to \lambda\tau, \ \mathbf{x} \to \lambda\mathbf{x}$$

1.4 Single-field slow roll inflation

We need a prolonged phase of accelerated expansion, with a background close to dS. The phase is called *inflation*.
However, the accelerated expansion given by dS solutions is eternal. Hence, this could not be connected with our universe. We need to introduce a clock ϕ that "turns off" Λ after some time (so that the dS phase can indeed stop when desired). We will take $\phi = \phi(t)$ to be a single, canonical scalar field, minimally coupled to gravity.

1.4.1 Prolonged quasi-de Sitter expansion

For inflation, we are postulating an early phase of accelerated expansion $\dot{a}, \ddot{a} > 0$. We reformulate it as:

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = H^2(1-\epsilon) > 0 \tag{1.36}$$

where we have defined the first Hubble slow-roll parameter

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \tag{1.37}$$

It is a dimensionless measure if the time variation of H. For a single-fluid universe, $\epsilon = 3(1+\omega)/2$: this follows from (1.27). And so we see that acceleration requires $\epsilon < 1 \iff \omega < -1/3$. We also want the Null Energy Condition (1.31) to be satisfied, so we need $\epsilon > 0 \iff \omega > -1$. Finally, since during inflation we want a background very close to dS spacetime ($\omega \approx -1$), we are interested in the regime $0 < \epsilon \ll 1$.

How long should inflation last? We define the number of *e-foldings* of expansion N by $dN \equiv Hdt$, which implies:

$$\Delta N = \Delta \log a \tag{1.38}$$

Based on the horizon problem, we need about 50 e-foldings during inflation: $\Delta N_{\text{infl}} \approx 50$. To see why, consult [5] or [1].
We generalise the definition of ϵ to higher order Hubble slow-roll parameters:

$$\xi_1 \equiv \epsilon \equiv -\frac{\dot{H}}{H^2} = -\partial_N \ln H \tag{1.39}$$

$$\xi_2 \equiv \eta \equiv -\frac{\dot{\epsilon}}{H\epsilon} = \partial_N \ln \epsilon \tag{1.40}$$

$$\xi_{n\geq 3} \equiv \partial_N \ln \xi_{n-1} \tag{1.41}$$

Taylor expanding ϵ around some reference N and requiring that it doesn't change much during inflation gives the conditions $\eta \Delta N_{\text{infl}}, \xi_n \Delta N_{\text{infl}} < 1$. (The derivation is in the Part III notes [5].) This motivates the *slow roll conditions* for the Hubble slow-roll parameters:

$$\boxed{\epsilon, \eta, \xi_n \ll 1 \text{ (slow roll inflation)}} \tag{1.42}$$

1.4.2 Single field inflation

We now want to ask how such an expansion history can emerge dynamically, from solving the equations of motion. To try to mimic a cosmological constant, we consider the action of scalar field coupled to gravity. Recalling the Hilbert-Einstein action (1.17), we find a minimally coupled, canonical scalar field ϕ with an action S:

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[M_{Pl}^2 R - \partial_\mu \phi \partial^\mu \phi - 2V(\phi) \right] \tag{1.43}$$

Here the potential $V(\phi)$ is an arbitrary function, so the we can read "$T - V$" in the Lagrangian (density). We consider the matter part of (1.43) and vary it with respect to the spacetime metric.

$$\delta S_M = \int d^4x \sqrt{-g} \left(\frac{1}{4} g_{\mu\nu} \nabla^\rho \phi \nabla_\rho \phi + \frac{1}{2} g_{\mu\nu} V(\phi) - \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi \right) \delta g^{\mu\nu} \tag{1.44}$$

The first two terms come from varying $\sqrt{-g}$. (I used (1.11)) while the final one comes from varying the metric in the gradient term $\partial_\mu \phi \partial^\mu \phi = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$. Recall that since ϕ is scalar, $\partial_\mu \phi = \nabla_\mu \phi$. We find the energy momentum tensor using it definition (1.19):

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V(\phi) \right] \tag{1.45}$$

This takes the same form as the energy-momentum tensor for a perfect fluid (recall (1.3)), under the identifications

$$\rho = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi) \tag{1.46}$$

$$p = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{1.47}$$

$$u_\mu = \frac{\partial_\mu\phi}{\sqrt{-\partial_\mu\phi\partial^\mu\phi}} \tag{1.48}$$

$$\tag{1.49}$$

If we specify $\phi = \phi(t)$, which seems reasonable by homogeneity, we get:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1.50}$$

$$p = -\frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{1.51}$$

$$u_\mu = (1, 0) \tag{1.52}$$

$$\tag{1.53}$$

Let's recall what we did in Part II Cosmology:

1. We also restricted to this homogeneous case.
2. There we also had (1.50) and (1.51). This makes me wonder: how did we obtain them at the time, without all the stuff about $T_{\mu\nu}$?
3. The answer is that we did <u>not</u> derive them. We just quoted those results.

To obtain the equations of motion, we vary the action (1.43) with respect to $\phi = \phi(t)$. (Or use the Euler-Lagrange equations). We set $K = 0$, because we are interested in accelerated expansion, which dilutes spatial curvature. In this case, $\sqrt{-g} = a^3$, and it's easy to obtain (see [1]):

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{1.54}$$

The first and last term are analogous to Newton's Second Law, while the middle term represents a genuinely relativistic effect. It's sometimes called *Hubble friction* and always opposes changes in ϕ, slowing down the field. The system is closed using the Friedmann equation (1.26), with ρ given as above by (1.50):

$$3H^2 M_{Pl}^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1.55}$$

The corresponding acceleration equation is :

$$-\dot{H} M_{Pl}^2 = \frac{1}{2}\dot{\phi}^2 \tag{1.56}$$

For almost any potential, these equations of motion cannot be solved exactly. Nevertheless, we can use the *Hamilton-Jacobi formalism* to obtain the right scalar potential that gives rise to some class of solution.

The Hamilton-Jacobi Formalism and exact solutions We can divide (1.56) by $\dot{\phi}$ to find that

$$2\frac{\partial H}{\partial \phi} M_{Pl}^2 = -\dot{\phi} \tag{1.57}$$

where $H = H(t(\phi))$. Then the Friedmann Equation (1.55) can be re-written as

$$3H^2 M_{Pl}^2 = 2M_{Pl}^4 \left(\frac{\partial H}{\partial \phi}\right)^2 + V(\phi) \tag{1.58}$$

One can then choose some function $H(\phi)$ and find the potential V from this algebraic equation (1.58). Then the first order differential equation (1.57) can be solved to find $\phi(t)$ and hence $H(t)$.
Notes:

1. We saw this method on the second example of Part II Cosmology (question 8).

2. Connection to Hamilton-Jacobi equation from Classical Dynamics?

Non-canonical scalar fields: A canonical scalar field has a simple quadratic term with one spacetime derivative per field, as in (1.56). There are more general, but still covariant options. Here we consider a generic function $P(X, \phi)$ of ϕ and the kinetic term $X \equiv -\partial_\mu \phi \partial^\mu \phi/2$. In this case, the relevant term in the action is $S_M = \int d^4x \sqrt{-g} P(X, \phi)$.
A similar calculation to (1.44), which we did for the specific case $P = X - V(\phi)$, gives us the energy-momentum tensor:

$$\delta S_M = \int d^4x \sqrt{-g} \frac{1}{2} \left(-g_{\mu\nu} P - P_X \nabla_\mu \phi \nabla_\nu \phi\right) \delta g^{\mu\nu} \tag{1.59}$$

and thus:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}P \tag{1.60}$$

So $p = P$ which implies $(\rho + P) = 2XP_X$ and $u_\mu = \partial_\mu\phi/\sqrt{2X}$, hence $\rho = 2XP_X - P$. So the Friedmann and acceleration equations are:

$$3M_{Pl}^2 H^2 = 2XP_X - P \tag{1.61}$$

$$-M_{Pl}^2 \dot{H} = XP_X \tag{1.62}$$

If we restrict to homogenous fields and use $\sqrt{-g} = a^3$, the Euler-Lagrange equation (for a variation $\phi \to \phi + \delta\phi$) gives us the equation of motion for ϕ:

$$\boxed{\ddot{\phi}(P_X + 2XP_{XX}) + 3H\dot{\phi}P_X + (2XP_{X\phi} - P_\phi) = 0} \tag{1.63}$$

Using (1.61) and (1.62), the Hubble slow roll parameter (1.39) is given by

$$\epsilon = \frac{3XP_X}{2XP_X - P} \tag{1.64}$$

1.4.3 Potential slow-roll parameters

The Hubble slow-roll parameters (1.39)- (1.41), together with the slow-roll condition (1.42), express in a compact way the requirements for an extended phase of inflation. However, they depend implicitly on ϕ. Given the some $V(\phi)$, one needs to solve the full dynamics to find $H(t)$.
We define the *potential slow-roll parameters* as:

$$\epsilon_V \equiv \frac{M_{Pl}^2}{2}\left(\frac{V'}{V}\right)^2 \tag{1.65}$$

$$\eta_V \equiv M_{Pl}^2 \frac{V''}{V} \tag{1.66}$$

$$\xi_{3V} \equiv M_{Pl}^4 \frac{V'V'''}{V^2} \tag{1.67}$$

It can be shown (see [5]) that provided that the slow roll condition for the Hubble slow-roll parameters is satisfied,

$$\boxed{\epsilon \approx \epsilon_V} \text{ and } \boxed{\eta \approx 4\epsilon_V - 2\eta_V} \tag{1.68}$$

However, for our purposes, the potential slow roll parameters won't be useful. We'll use the Hubble slow roll parameters, and their slow-roll condition (1.42).

1.4.4 Slow-roll inflation

The assumption that the slow-roll parameters are small allows us to find approximate solutions to the EOM. Using

$$X = \frac{\dot{\phi}^2}{2}$$

, $\rho = X + V$, $p = X - V$, we can rewrite the continuity equation (1.24) as

$$\dot{X} + 6HX + V'\dot{\phi} = 0 \tag{1.69}$$

which is equivalent to (1.63) multiplied by $\dot{\phi}$ (which we'll assume non-zero). Making use of the condition $\epsilon \ll 1$, we have

$$\begin{aligned} 1 &\gg \epsilon \\ &= -\frac{\dot{H}}{H^2} \text{ by definition (1.39)} \\ &= \frac{X}{M_{Pl}^2 X^2} \text{ by the acceleration equation (1.56)} \\ &= \frac{3X}{V+X} \text{ by the Friedmann equation (1.55)} \end{aligned}$$

Hence, $X = \frac{\dot{\phi}^2}{2} \ll V$, which is one of the slow-roll condition in the form we saw in Part II Cosmology. This also implies

$$\boxed{3M_{Pl}^2 H^2 \approx V} \tag{1.70}$$

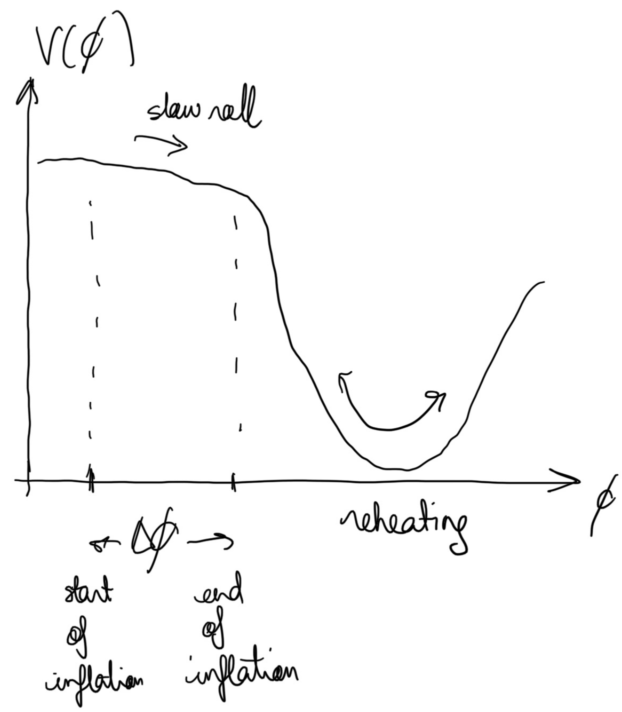
We can also easily derive the <u>exact</u> relation $\eta = 2\epsilon + \frac{\dot{X}}{XH}$. Since $\epsilon, \eta \ll 1$, we learn that $\dot{X} \ll XH$ and hence $2\ddot{\phi} \ll \dot{\phi}H$. This is the second slow-roll condition in the form from Part II Cosmology. It tells us that we can neglect the acceleration term $\ddot{\phi}$ in (1.63) and thus:

$$\boxed{3H\dot{\phi} \approx -V'} \tag{1.71}$$

Combining both approximations, one can reduce the problem to

$$\dot{\phi} \approx -\frac{V' M_{Pl}}{\sqrt{3V}} \implies \boxed{t = \int d\phi \frac{\sqrt{3V}}{V' M_{Pl}}} \tag{1.72}$$

The resulting $\phi(t)$ is the slow-roll solution, which is a good approximation to



the exact solution when $\epsilon, \eta \ll 1$

1.4.5 End of inflation and reheating

By definition, inflation ends at t_e where $\epsilon(t_e) \geq 1$, so that the expansion starts to decelerate. (In the slow-roll approximation, this means $\epsilon_V(\phi_e) \approx 1$, where $\phi_e = \phi(t_e)$. Via the chain rule, one can estimate ([5]):

$$\frac{\Delta\phi}{M_{Pl}} \approx \Delta N_{\text{infl}} \frac{M_{Pl} V'}{V} \tag{1.73}$$

where $\Delta\phi = |\phi_e - \phi_i|$.

As the inflaton oscillates around the minimum of the potential, with ever decreasing amplitude due to the Hubble friction term in (1.63), quantum processes become relevant and the inflaton decays into a hot soup of standard model particles.

Reheating [1], [6]: By the end of inflation, the universe us left flat but devoid of any matter and radiation. For this to be a realistic mechanism, we must find a way to transfer energy from the inflaton field into more traditional forms of matter. This is done by coupling the inflaton field to standard model fields. As the inflaton oscillates around the minimum of its potential, these other fields become excited. This process is known as *reheating* Afterwards, the standard hot Big Bang cosmology can start.

Issues with inflation

- What is ϕ?
- Is V natural?
- Initial conditions: we must start with ϕ sitting high in the potential. How did it get there?
- Measure Problem: Given a theory, how likely is inflation? Of all the possible positions we could begin with, how likely is that we start with a ϕ_i and $\dot{\phi}_i$ which allow inflation?

2 Free Quantum Field Theory

This section is very heavily based on the David Tong's notes for Part III QFT [9]. They are also based in [16].

Conventions for this section: Natural units with $\hbar = 1 = c$ are used in this section (as in the previous one). However, now we use the mostly minus convention (+,-,-,-). For vectors, I use both an arrow ($\vec{v}$) or bold font ($\mathbf{v}$).

2.1 Classical Field Theory

2.1.1 The Dynamics of Fields

A *field* is a quantity defined at every point of space and time $(\vec{x}, t)$. In field theory we're interested in the dynamics of fields $\phi_a(\vec{x}, t)$ where both $\vec{x}$ and a are labels.

- Eg, the Electromagnetic Field. In particular, recall the 4-component field $A^\mu = (\phi, \vec{A})$ in *spacetime* (see eg, [4]).
- Eg, the inflaton field ϕ we used in the previous section to model inflation.

For the systems that we'll consider, the dynamics of the field is governed by a Lagrangian (density) $\mathcal{L}$ obeying the Euler-Lagrange equations of motion:

$$\boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) = \frac{\partial \mathcal{L}}{\partial \phi_a}} \tag{2.1}$$

An Example: The Klein Gordon Equation The Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) \tag{2.2}$$

for the real scalar field ϕ gives the *Klein-Gordon* (KG) equation

$$\boxed{\partial_\mu \partial^\mu \phi + m^2 \phi = 0} \tag{2.3}$$

as equation of motion. Note:

- We are contracting indices with the Minkowski metric $\eta^{\mu\nu}$ (instead of with $g^{\mu\nu}$, which we used in the previous section where there was a gravitational field!)
- This Lagrangian is Lorentz invariant, and so is the KG equation.
- We can generalise this to a Lagrangian with a general potential $V(\phi)$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \implies \boxed{\partial_\mu\partial^\mu\phi + \frac{\partial V}{\partial\phi} = 0} \tag{2.4}$$

  - This is the canonical Lagrangian that we used in inflation (1.43)
- We can write a similar Lagrangian for a complex scalar field $\psi(x)$:

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi \implies \boxed{\partial_\mu\partial^\mu\psi + m^2\psi = 0} \tag{2.5}$$

  after treating treat ψ and ψ^* as independent objects.

Another Example: First Order Lagrangians Consider a complex scalar field ψ whose dynamics is given by the real Lagrangian

$$\mathcal{L} = im(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \nabla\psi^*\nabla\psi - 2m^2\psi^*\psi \tag{2.6}$$

Note that it's <u>not</u> Lorentz invariant.
The equation of motion is then

$$\boxed{i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi + m\psi} \tag{2.7}$$

-This is very similar to the (time-dependent) Schrödinger Equation. But ψ is classical, there's no probabilistic interpretation of this equation! We'll see more of this when we quantize fields in the next subsection.
For our purposes, this example is not very relevant because we only care about real fields.

A Final Example: Maxwell's Equations In vacuum, the Lagrangian for the EM fields is

$$\mathcal{L} = \frac{1}{2}[-(\partial_\mu A_\nu)(\partial^\mu A^\nu) + (\partial_\mu A^\mu)^2]$$

Note that it is Lorentz invariant. It gives Maxwell's equations is vacuum:

$$\boxed{-\partial_\mu F^{\mu\nu} = 0} \tag{2.8}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. In Electrodynamics [4] we have a more general Lagrangian giving the sourced Maxwell equations, $-\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$.

2.1.2 Lorentz Invariance

- The Laws of Nature are relativistic. We want to construct field theories in which space and time are placed on an equal footing, and which is Lorentz invariant.
- A theory is *Lorentz invariant* if whenever $\phi(x)$ solves the equations of motion, so does $\phi(\Lambda^{-1}x)$, for any Lorentz transformation Λ in the Lorentz group.
- We can ensure that this holds by ensuring that the action $S = \int d^4x\mathcal{L}$ is Lorentz invariant.
- Examples (from above): the KG equation (2.3) and the Maxwell's equations (2.8) are Lorentz invariant, while the EoM for the first order dynamics (2.7) is not.

2.1.3 Symmetries

Noether's Theorem Let's recall *Noether's Theorem* (from Classical Dynamics [11]):
Every continuous symmetry of the Lagrangian gives rises to a conserved *current* $j^\mu(x)$, such that the equations of motion imply $\partial_\mu j^\mu = 0$. Moreover, this implies a conserved charge

$$Q = \int d^3x j^0$$

The current is given by

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} X_a(\phi) - F^\mu(\phi) \tag{2.9}$$

where the continuous symmetry is defined infinitesimally by $\delta\mathcal{L} = \partial_\mu F^\mu$

An Example: Translations and the Energy-Momentum Tensor In classical particle mechanics:

- Invariance under spatial translations gives rise to the conservation of momentum.
- Invariance under time translations gives rise to the conservation of energy.

In field theories, we will see something similar. Consider the *active* infinitesimal transformation given by $x^\mu \to \tilde{x}^\mu \equiv x^\mu - \epsilon^\mu$. Taylor expanding then gives:

$$\phi_a(x) \to \phi_a(x(\tilde{x})) = \phi_a(\tilde{x} + \epsilon) = \phi_a(\tilde{x}) + \epsilon^\nu \frac{\partial}{\partial \tilde{x}^\nu} \phi_a(\tilde{x}) \tag{2.10}$$

$$\mathcal{L}(x) \to \mathcal{L}(x(\tilde{x})) = \mathcal{L}(\tilde{x} + \epsilon) = \mathcal{L}(\tilde{x}) + \epsilon^\nu \frac{\partial}{\partial \tilde{x}^\nu} \mathcal{L}(\tilde{x}) \tag{2.11}$$

We will now drop the tilde, so that $\mathcal{L} =\rightarrow \mathcal{L} + \epsilon^\nu \partial_\nu \mathcal{L}$. Since the Lagrangian is a total derivative, we may use Noether's theorem. It gives us four (independent) conserved currents $(j^\mu)_\nu$ for each of the translation ϵ^ν with $\nu = 0, 1, 2, 3$:

$$\boxed{(j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L} \equiv T^\mu_{\ \nu}} \tag{2.12}$$

$T^\mu_{\ \nu}$ is called the *energy-momentum tensor*, as in the previous section! It satisfies $\partial_\mu T^\mu_{\ \nu} = 0$ The four conserved quantities are $E = \int d^3x T^{00}$ and $P^i = \int d^3x T^{0i}$. These are the total energy and the total momentum of the field configuration, respectively.

- **An Example of the Energy-Momentum Tensor:** Consider the simplest scalar field theory with Lagrangian

  $$\mathcal{L} = \frac{1}{2}(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2)$$

  It has $T^{\mu\nu} = \partial^\nu\phi\partial^\mu\phi - \eta^{\mu\nu}\mathcal{L}$, which is symmetric. This won't always be the case.

– The conserved energy and momentum are given by:

$$E = \int d^3x \frac{1}{2}\left(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right) \tag{2.13}$$

$$P^i = \int d^3x \dot{\phi}\partial^i\phi \tag{2.14}$$

– There's a way to obtain a symmetric conserved form. We can write

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho\Gamma^{\rho\mu\nu}$$

where $\Gamma^{\rho\mu\nu} = -\Gamma^{\mu\rho\nu}$ is chosen so that $\Theta^{\mu\nu} = \Theta^{\nu\mu}$. This also implies that $\Theta^{\mu\nu}$ is a conserved current.

– One reason that makes you want a symmetric energy-momentum tensor is to make contact with general relativity. Firstly, consider coupling the theory to a curved background spacetime, introducing an arbitrary metric $g_{\mu\nu}(x)$ in place of $\eta_{\mu\nu}$: $d^4x \to d^4x\sqrt{-g}$ The replace the kinetic terms with suitable covariant derivatives using "minimal coupling": $\eta^{\mu\nu}\partial_\nu\phi\partial_\mu\phi \to g^{\mu\nu}\nabla_\nu\phi\nabla_\mu\phi$. Then a symmetric energy momentum tensor in the flat space theory is given by

$$\Theta^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}}\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}} \tag{2.15}$$

Another Example: Lorentz Transformation and Angular Momentum

Internal Symmetries

2.1.4 The Hamiltonian Formalism

We need the Hamiltonian formalism of field theory to make connections with quantum theory.
We define the *momentum* $\pi^a(x)$ conjugate to the field $\phi_a(x)$ by

$$\pi^a(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_a} \tag{2.16}$$

Similarly to classical mechanics (see [11]), we defined the *Hamiltonian density* by

$$\mathcal{H}(\phi_a(x), \pi^a(x); x) = \pi^a(x)\dot{\phi}_a(x) - \mathcal{L} \tag{2.17}$$

and the Hamiltonian is

$$H = \int d^3x \mathcal{H}$$

- Eg, for the Lagrangian $\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)$, the conjugate momentum is $\pi = \dot{\phi}$, which gives the Hamiltonian,

$$H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right)$$

In the Hamiltonian formalism, the equations of motion for $\phi(x) = \phi(\vec{x}, t)$ arise from *Hamilton's equations*:

$$\boxed{\dot{\phi}(\vec{x}, t) = \frac{\partial\mathcal{H}}{\partial\pi(\vec{x}, t)}} \text{ and } \boxed{\dot{\pi}(\vec{x}, t) = -\frac{\partial\mathcal{H}}{\partial\phi(\vec{x}, t)}} \tag{2.18}$$

These equations can be derived in the same way is in classical mechanics [11].

2.2 Free Fields

From now on, we assume knowledge of Part II Principles of Quantum Mechanics [12]. In particular, we use: Dirac notation, ladder operators for the harmonic oscillator, the Heisenberg picture.

2.2.1 Canonical Quantization

A *quantum field* is an <u>operator</u> valued function of <u>space</u> obeying the commutation relations

$$[\phi_a(\mathbf{x}), \phi_b(\mathbf{y})] = [\pi^a(\mathbf{x}), \pi^b(\mathbf{y})] = 0 \tag{2.19}$$

$$[\phi_a(\mathbf{x}), \pi^b(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})\delta_a^b \tag{2.20}$$

Comments:

- We are working in the Schrödinger picture so that the operator $\phi_a(\mathbf{x})$ and $\pi^a(\mathbf{x})$ do not depend on time at all - only on space. (All time dependance sits in states of the Hilbert space evolving according to the usual TDSE).

- If you were to write the wavefunction in QFT, it would be a *functional*: a function of every possible configuration of the field ϕ.

The typical information we want to know about a quantum theory is the spectrum of the Hamiltonian H. In QFTs this is usually very hard because we have an infinite number of degrees of freedom. Nonetheless, for *free theories* we can find a way to write the dynamics such that each degree of freedom evolves independently from all the others.
The simplest relativistic free theory is the classical Klein-Gordon (KG) (2.3) equation for a real scalar field $\phi(\mathbf{x}, t)$. After taking a Fourier transform over space, we find that for each value of $\mathbf{p}$, $\tilde{\phi}(\mathbf{p}, t)$ [2] solves the equation of a harmonic oscillator vibrating at frequency

$$\boxed{\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}} \tag{2.21}$$

Hence, the most general solution to the KG equation is a linear superposition of simple harmonic oscillators, each vibrating at a different frequency with a different amplitude. We will quantize this.

2.2.2 The Free Scalar Field

To quantize the Klein–Gordon field, we just have to quantize this infinite number of harmonic oscillators!

We are going to do this in two steps. First, we write our quantum fields $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ in terms of their Fourier transforms

$$\begin{aligned}
\phi(\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}) \\
\pi(\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \pi(\mathbf{p})
\end{aligned}$$

Confusingly, π represents both the conjugate momentum and the mathematical constant, but it should be clear from the context.

If we believe in our classical analogy, then the operators $\phi(\mathbf{p})$ and $\pi(\mathbf{p})$ should represent the position and momentum of quantum harmonic oscilla-

[2] It is very common to abuse notation by writing just $\phi(\mathbf{p}, t)$

tors. So we further write them as

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{2.22}$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\right), \tag{2.23}$$

where we have

$$\omega_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2.$$

Note that despite what we said above, we are multiplying $a^\dagger_{\mathbf{p}}$ by $e^{-i\mathbf{p}\cdot\mathbf{x}}$, and not $e^{i\mathbf{p}\cdot\mathbf{x}}$. This is so that $\phi(\mathbf{x})$ will be manifestly a real quantity.
We have:

$$\phi(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right) \tag{2.24}$$

$$\pi(\mathbf{p}) = -i\sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right) \tag{2.25}$$

so

$$a_{\mathbf{p}} = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (\omega_{\mathbf{p}} \phi(\mathbf{p}) + i\pi(\mathbf{p})) \tag{2.26}$$

which the analogous form to the usual operator a for the harmonic oscillator in QM.

Commutation relations and Hamiltonian It can be shown that the commutation relations (2.19), (2.20) for ϕ and π as given by (2.22) and (2.23) are equivalent to the following for $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$:

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0 = [a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] \text{ and } [a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{2.27}$$

Note: These commutator relationships (2.27) together with (2.24) and (2.25) imply:

$$[\phi(\mathbf{p}), \phi(\mathbf{q})] = 0 \tag{2.28}$$

$$[\pi(\mathbf{p}), \pi(\mathbf{q})] = 0 \tag{2.29}$$

$$[\phi(\mathbf{p}), \pi(\mathbf{q})] = (2\pi)^3 i\delta^{(3)}(\mathbf{p} + \mathbf{q}) \tag{2.30}$$

which differ only from the real space commutator relationships in the sign inside the delta function of the last line.
One can also compute the Hamiltonian in terms of $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$:

$$H = \frac{1}{2}\int d^3x \left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right) \tag{2.31}$$

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}} \left(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) \tag{2.32}$$

We have skipped many steps. To derive this, we've used the expressions (2.22) and (2.23) for the field, the identity

$$(2\pi)^3\delta^{(3)}(0) = \int d^3x \exp\{i\mathbf{k}\cdot\mathbf{x}\}$$

and the definition (2.21) of $\omega_{\mathbf{p}}$.
Using the commutation relations (2.27), we would get

$$H = \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2}(2\pi)^3\delta^{(3)}(0)\right)$$

We have a problem here. This diverges, and for two reasons:

1. Inside the integral, we have a delta function evaluated at the origin, where it is infinitely large.

2. The integral over $\omega_{\mathbf{p}}$ diverges at large $\|\mathbf{p}\|$

We'll see how to solve this in the next section.

2.2.3 The Vacuum

Analogously to the 1-d harmonic oscillator, we define the *vacuum* state $|0\rangle$ by requiring that it is annihilated by <u>all</u> $a_{\mathbf{p}}$:

$$a_{\mathbf{p}}|0\rangle = 0 \; \forall\mathbf{p} \tag{2.33}$$

Without going into details here, the infinity that we see with the Hamiltonian as above corresponds to the energy E_0 of the vacuum. However, in physics, we are interested in energy differences (there's no way to measure E_0 directly[3]. So we can redefine the Hamiltonian by subtracting off this infinity, so we write

$$H \equiv \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \tag{2.34}$$

With this definition, $H|0\rangle = 0$.
This method is called *normal ordering*, and we will formalise as follows: We write the *normal order* string of operators $\phi_1(\vec{x_1}) \dots \phi_n(\vec{x_n})$ as

$$: \phi_1(\vec{x_1}) \dots \phi_n(\vec{x_n}) : \tag{2.35}$$

with all annihilation operators $a_{\vec{p}}$ placed to the right.

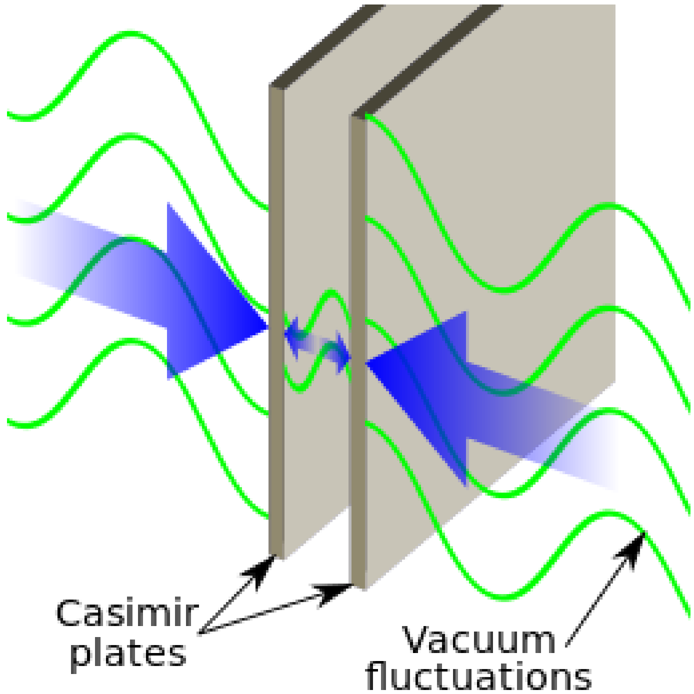
- Eg, $: a^\dagger_{\mathbf{p}} a_{\mathbf{p}} := a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$
- Eg, $: a_{\mathbf{p}} a^\dagger_{\mathbf{p}} := a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$
- Eg, $: a_{\mathbf{p}} a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{p}} := a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$
- Eg, if we take H as in (2.32),

  $$: H := \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$$

  So we recover (2.34), which is what we wanted.

The Casimir Effect [10]

[3]There is a big caveat here: gravity is supposed to see everything! The sum of all zero point energies should contribute to the stress-energy tensor that appears on the Einstein's Equations (1.21) as the a cosmological constant Λ. Current observations suggest that $\Lambda \approx O((10^{-3} eV)^4)$, which is much smaller than other scales in particles physics. This is the cosmological constant : why don't the zero point energies of these fields contribute to Λ? Or, if they do, what cancels them to such high accuracy?



There is a situation in where the differences in the energy of vacuum fluctuations themselves can be measured. These are physical forces arising from a quantized field. One simple case is that of two uncharged plates in vacuum, which will be attracted to each other.

2.2.4 Particles

Having dealt with the vacuum, we can now turn to the excitations of the field. It's easy to verify that

$$[H, a_{\mathbf{p}}^{\dagger}] = \omega_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}$$

and

$$[H, a_{\mathbf{p}}] = -\omega_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}$$

This means that, as for the harmonic oscillator, we can construct energy eigenstates by acting on the vacuum $|0\rangle$ with $a_{\mathbf{p}}^{\dagger}$.
We let

$$|\mathbf{p}\rangle = a_{\mathbf{p}}^{\dagger} |0\rangle$$

so that this states has energy $\omega_{\mathbf{p}}$, i.e.

$$H\,|\mathbf{p}\rangle = \omega_{\mathbf{p}}\,|\mathbf{p}\rangle$$

Since $\omega_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2 \equiv E_{\mathbf{p}}^2$, we interpret $|\mathbf{p}\rangle$ as the momentum eigenstate of a single particle of mass m.
We promote the classical total momentum $\mathbf{P}$, given in (2.14), and promote it to an operator:

$$\mathbf{P} = -\int d^3x \pi \nabla\phi = \int \frac{d^3p}{(2\pi)^3}\mathbf{p} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \tag{2.36}$$

The proving the second equality is an exercise in Example Sheet 2 of the QFT course. Acting on our state $|\mathbf{p}\rangle$ with $\mathbf{P}$, we see that it is indeed a momentum eigenstate:

$$\mathbf{P}\,|\mathbf{p}\rangle = \mathbf{p}\,|\mathbf{p}\rangle$$

One can also show (in the second example sheet as well) that if we promote the classical total angular momentum to an operator $\vec{J}$, then one-particle state with zero momentum carries no internal angular momentum:

$$\vec{J}\,|\mathbf{p} = 0\rangle = 0$$

In other words, quantizing a scalar field gives rise to a spin-0 particle.
Some more things to note:

- **Multi-Particle States, Bosonic Statistics, Fock Space, Number Operator**
  - We create multi-particle states by acting multiple times with $a^\dagger$'s. We interpret
    $$|\vec{p}_1, \ldots, \vec{p}_n\rangle \equiv a_{\vec{p}_1}^\dagger \ldots a_{\vec{p}_n}^\dagger\,|0\rangle \tag{2.37}$$
    as an *n-particle state.*
  - These particles are *bosons*, because all the creation operators commute among themselves.
  - The full Hilbert state of our theory is spanned by acting on the vacuum with all possible combinations of $a^\dagger$'s, and it is called the *Fock space.*

- The *number operator* N is

$$N = \int \frac{d^3p}{(2\pi)^3} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$$

counts the number of particles in a given state.
It commutes with the Hamiltonian, so the particle number is conserved.

- **Operator Valued Distributions:**
  - We are calling the $|\mathbf{p}\rangle$ as "particles", but they really aren't. They are *momentum eigenstates*.
  - Theoretically, we can create a localized state via a Fourier transform:

$$|\mathbf{x}\rangle = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} |\mathbf{p}\rangle \tag{2.38}$$

  - In QM, neither $|\mathbf{x}\rangle$ nor $|\mathbf{p}\rangle$ are good elements of the Hilbert space because they are not normalizable. Similarly, in QFT neither $\phi(\mathbf{x})$ nor $a_{\mathbf{p}}$ are good operators acting on the Fock space, because they don't produce normalizable states:

$\langle 0| a_{\mathbf{p}} a_{\mathbf{p}}^\dagger |0\rangle = \langle \mathbf{p}|\mathbf{p}\rangle = (2\pi)^3 \delta^{(3)}(0)$ and $\langle 0| \phi(\mathbf{x})\phi(\mathbf{x}) |0\rangle = \langle \mathbf{x}|\mathbf{x}\rangle = \delta(0)$

  - We can construct well defined operators by smearing these distributions over space:

$$|\varphi\rangle = \int \frac{d^3p}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{x}} \hat{\varphi}(\mathbf{p}) |\mathbf{p}\rangle$$

Relativistic Normalization Some facts...

- We have defined the vacuum $|0\rangle$, which we normalise as $\langle 0|0\rangle = 1$.
- The Lorentz invariant measure is

$$\int \frac{d^3p}{2E_{\mathbf{p}}} \tag{2.39}$$

- The identity operator on one-particle states is

$$I = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \langle p|p\rangle \tag{2.40}$$

where the relativistically normalized momentum is $|p\rangle = \sqrt{2E_{\mathbf{p}}} |\mathbf{p}\rangle$.

2.2.5 Complex Scalar Fields

Not a particularly relevant section for our purposes.
We consider a complex scalar field field $\psi(x)$ obeying the KG equation: its dynamics is given by (2.5). We expand the corresponding quantum (non-hermitian) field operator as a sum of plane waves

$$\psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(b_{\mathbf{p}} e^{+i\mathbf{p}\cdot x} + c_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \tag{2.41}$$

with its conjugate momentum $\pi = \partial\mathcal{L}/\partial\dot{\psi} = \dot{\psi}^*$ given by

$$\pi = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\mathbf{p}}}{2}} \left(b_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} - c_{\mathbf{p}} e^{+i\mathbf{p}\cdot\mathbf{x}} \right) \tag{2.42}$$

The commutation relations are:

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \text{ and } [\phi(\mathbf{x}), \pi^\dagger(\mathbf{y})] = 0$$

together with $[\phi(\mathbf{x}), \phi(\mathbf{y})] = 0 = [\phi(\mathbf{x}), \phi(\mathbf{y})^\dagger]$.
These are equivalent to

$$[b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{2.43}$$

$$[c_{\mathbf{p}}, c_{\mathbf{q}}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{2.44}$$

$$[b_{\mathbf{p}}, b_{\mathbf{q}}] = 0 \tag{2.45}$$

$$[b_{\mathbf{p}}, c_{\mathbf{p}}^\dagger] = 0 \tag{2.46}$$

$$[b_{\mathbf{p}}, c_{\mathbf{p}}] = 0 \tag{2.47}$$

and so on.
Quantizing a complex scalar field gives rise to two creation operators: $b_{\mathbf{p}}^\dagger, c_{\mathbf{p}}^\dagger$- They create particles and antiparticles, respectively, both of mass m and spin zero. After normal ordering, the conserved charge is the quantum operator given by

$$Q = \int \frac{d^3p}{(2\pi)^3} (c_{\mathbf{p}}^\dagger c_{\mathbf{p}} - b_{\mathbf{p}}^\dagger b_{\mathbf{p}}) \equiv N_c - N_b \tag{2.48}$$

which counts the number of anti-particles minus the number of particles.

2.2.6 The Heisenberg Picture

Most of QM is done in the Schrödinger picture. But we want QFT to be relativistic, so we need time and space to be in a similar footing. Hence, in QFT, the Heisenberg picture is the most used in practice.
In field theory, we distinguish between the Schrödinger and the Heisenberg picture by specifying the dependence of the fields:

- **Schrödinger picture** The fields (which are operators!) depend only on space: $\phi = \phi(\vec{x})$
- **Heisenberg picture** Now the states are fixed in time, but the fields depend on spacetime: $\phi = \phi(x) = \phi(\vec{x}, t)$

In the Heisenberg picture, one can check that the equation of motion are:

$$\dot{\phi} = i[H, \phi] = \pi(x) \tag{2.49}$$

$$\dot{\pi} = i[H, \pi] = \nabla^2\phi - m^2\phi \tag{2.50}$$

In particular, ϕ satisfies the Klein-Gordon equation (2.3).
In the Heisenberg picture, operators $\mathcal{O}$ satisfy

$$\mathcal{O}_H = e^{iHt}\mathcal{O}_S e^{-iHt}$$

so

$$\phi(x) = e^{iHt}\phi(\vec{x})e^{-iHt}$$

which, after playing with commutators, gives the expression for ϕ in the Heisenberg picture.

$$\boxed{\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)} \tag{2.51}$$

where $p \cdot x = p_\mu x^\mu = E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x}$ (which explains the opposite sign in the exponents of e).

Causality For our theory to be causal, we must require that all spacelike separated operators commute. This ensures that a measurement at x cannot affect a measurement at y when x and y are not causally connected. One can show that

$$\Delta(x-y) \equiv [\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ip\cdot(x-y)} + e^{ip\cdot(x-y)}\right) \tag{2.52}$$

which, furthermore:

- Is a Lorentz invariant function.
- Doesn't vanish for timelike separations.
- Vanishes for spacelike separations.

So our theory is indeed casual with commutators vanishing outside the lightcone.

- This is really all we need for causality, it doesn't matter that $D(x-y) \neq 0$ outside the lightcone.
- This is because only commutators tell us anything about causality (from the Uncertainty Principle: if the commutator is zero, then in theory we can reduce the variance of one observable without necessarily augmenting the variance of the other.)
- Indeed, we define a theory to be *causal* if for any space-like separated points x, y, and any two fields ϕ, ψ, we have
$$[\phi(x), \psi(y)] = 0.$$

2.2.7 Propagators

For x and y spacetime points, we define the *propagator* $D(x-y)$ as the amplitude to find at x a particle prepared at y:

$$\langle 0| \phi(x)\phi(y) |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)} \equiv D(x-y) \qquad (2.53)$$

For spacetime separations, it decays[4] as $D(x-y) \sim e^{-m\|\vec{x}-\vec{y}\|}$. So it decays exponentially quickly outside the lightcone, but nonetheless, it's non-vanishing! Some things to note

- $[\phi(x), \phi(y)] = D(x-y) - D(y-x) = 0$ if $(x-y)^2 < 0$. If a particle can travel in a spacelike direction from x to y, it can also do so in the opposite way, from y to x. Hence, in any measurement, the amplitudes for these two events cancel.
- With a complex scalar field, $[\psi(x), \psi(y)^\dagger] = 0$ outside the lightcone. The interpretation is that the amplitude for the particle to propagate from x to y cancels the amplitude for the antiparticle to travel from y to x. (For a real scalar field, the particle is its own antiparticle).

[4] For a derivation see page 30 of [13]

The Feynman Propagator The *Feymann propagator* is

$$\Delta_F(x-y) \equiv \langle 0| T\phi(x)\phi(y) |0\rangle \tag{2.54}$$

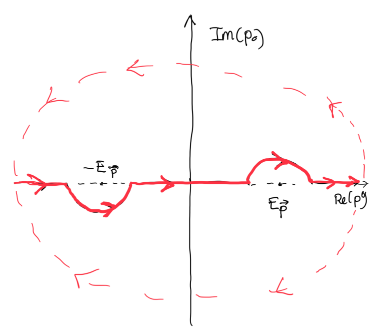
where T is the *time ordering* operator, placing all operators evaluated at later times to the left, so:

$$T\phi(x)\phi(y) \begin{cases} \phi(x)\phi(y) & x^0 > y^0 \\ \phi(y)\phi(x) & y^0 > x^0 \end{cases} \tag{2.55}$$

Note this is similar to the time ordered exponentials we met in PQM ([12]) to study time-dependent perturbations!
After contour integration, one can obtain an expression for the Feymann propagator in terms of a 4-momentum integral:

$$\boxed{\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} e^{-ip\cdot(x-y)}} \tag{2.56}$$

To evaluate this integral, one needs to use the Feymann contour:



where we choose to close in the upper or lower half plane in a way that allows us to use Jordan's lemma.

Green's Functions The Feymann propagator is also the Green's function for the Klein-Gordon operator. If we stay away from the singularities, we have:

$$\boxed{(\Box + m^2)\Delta_F(x-y) = -i\delta^{(4)}(x-y)} \tag{2.57}$$

Had we used a more traditional contour to avoid the poles, we would have obtained the familiar *Retarded* Green's function (from ED [4]) or the *Advanced* Green's function (from scattering in AQM [14]) .

2.2.8 Non-Relativistic Fields

This subsubsection is not really of our interests, because in Cosmology everything is relativistic! But it's cool. And a bit tricky. (And I've already typed the notes for it).

Let's return now to our classical complex scalar obeying the Klein-Gordon equation (2.5):

$$\psi(\vec{x},t) = e^{-imt}\tilde{\psi}(\vec{x},t) \implies \ddot{\tilde{\psi}} - 2im\dot{\tilde{\psi}} - \nabla^2\tilde{\psi} = 0$$

The non-relativistic limit is[5]

$$\|\vec{p}\| \ll m \implies \left|\ddot{\tilde{\psi}}\right| \ll m\left|\dot{\tilde{\psi}}\right|$$

And hence in the limit the KG equation is

$$\boxed{i\frac{\partial\tilde{\psi}}{\partial t} = -\frac{1}{2m}\nabla^2\tilde{\psi}} \tag{2.58}$$

1. This looks very much like the SE for a (non-relativistic) free particle of mass m.
2. Except it doesn't have any probability interpretation -it's simply a classical field evolving through an equation that's first order in time derivatives.
3. We saw the something similar in (2.7) when we considered a first order Lagrangian (2.6). (In fact, equivalent after substituting $\psi = e^{-imt}\tilde{\psi}$.)
4. Moreover, we will derive that first order Lagrangian (2.6) from the KG complex Lagrangian (in the non-relativistic limit).

After taking the non-relativistic limit, the complex KG Lagrangian of (2.5) becomes

$$\mathcal{L} = +im(\tilde{\psi}^*\dot{\tilde{\psi}} - \dot{\tilde{\psi}}^*\tilde{\psi}) - \nabla\tilde{\psi}^*\nabla\tilde{\psi} \tag{2.59}$$

[5] For a proof, see the answer given here:
https://www.physicsforums.com/threads/question-about-non-relativistic-limit-of-qft.709980/

-If we now invert the transformation, using $\tilde{\psi} \to e^{+imt}\psi$, the non-relativistic Lagrangian (2.59) becomes the first-order Lagrangian (2.6).
Back with (2.59), dropping the tildes $\tilde{(\cdot)}$ and dividing by $2m$ gives

$$\boxed{\mathcal{L} = +\frac{i}{2}(\psi^* \dot{\psi} - \dot{\psi}^* \psi) - \frac{1}{2m}\nabla\psi^*\nabla\psi} \tag{2.60}$$

The Hamiltonian density corresponding to (2.60) is

$$\boxed{\mathcal{H} = \frac{1}{2m}\nabla\psi^*\nabla\psi} \tag{2.61}$$

This is for classical field theory. To quantize, we impose

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}}$$

with

$$[a_{\vec{p}}, a^\dagger_{\vec{p}}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$$

The vacuum and the excitations are as before (2.37) The one-particle states $|\vec{p}\rangle$ have energy $\|\vec{p}\|^2/2m$, i.e.

$$\boxed{H\,|\vec{p}\rangle = \frac{\|\vec{p}\|^2}{2m}\,|\vec{p}\rangle}$$

which is the non-relativistic dispersion relation.
Quantizing the complex Lagrangian (2.5) gives rise to non-relativistic particles of mass m.

- The existence of anti-particles is a consequence of relativity: in the non-relativistic limit, despite having a complex field, we only have a single type of particle (the anti-particle is not in the spectrum).

- The conserved charge $Q = \int d^3x : \psi^\dagger \psi :$ is the particle number. Only with relativity can the particle number change.

- There is no non-relativistic limit of a real scalar field, because the particles are their own anti-particles.

Recovering Quantum Mechanics We want to recover the familiar results from Quantum Mechanics (which is non-relativistic - that's why do it in this section!).
We have (2.36) for the operator for the total angular momentum of the field

$$\boxed{\vec{P} = \int \frac{d^3p}{(2\pi)^3} \vec{p} a_{\vec{p}}^\dagger a_{\vec{p}}}$$

In the non-relativistic limit, the operator

$$\psi^\dagger(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}$$

creates a particle with a δ-function d at $\vec{x}$.
Why?: So that if we write $|\vec{x}\rangle = \psi^\dagger(\vec{x})|0\rangle$ we recover the (inverse FT) expression for $|x\rangle$, (2.38).

For the position operator, we can write

$$\boxed{\vec{X} = \int d^3x \vec{x} \psi^\dagger(\vec{x}) \psi(\vec{x})} \tag{2.62}$$

so that $\vec{X}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$.

We construct a state by taking superposition of one-particle states $|\vec{x}\rangle$ as

$$|\varphi\rangle = \int d^3x \varphi(\vec{x}) |\vec{x}\rangle$$

This implies that

$$X^i |\varphi\rangle = \int d^3x x^i \psi(\vec{x}) |\vec{x}\rangle \tag{2.63}$$

$$P^i |\varphi\rangle = \int d^3x \left(-i\frac{\partial \varphi}{\partial x^i}\right) |\vec{x}\rangle \tag{2.64}$$

$$[X^i, P^j] |\varphi\rangle = i\delta^{ij} |\varphi\rangle \tag{2.65}$$

Moreover, from the TDSE for $|\varphi\rangle$

$$i\frac{\partial |\varphi\rangle}{\partial t} = H |\varphi\rangle$$

we obtain

$$\boxed{i\frac{\partial\varphi}{\partial t} = -\frac{1}{2m}\nabla^2\varphi}$$

which is the same equation obeyed by the original field (2.58). Now it does have the usual probabilistic interpretation for the wavefunction φ: it is really the Schrödinger equation! The corresponding conserved charge (from the Noether current) is $Q = \int d^3x|\varphi(\vec{x})|^2$, it is the total probability.

2.3 A Interacting Fields

2.3.1 Introduction

The free field theories we have so far considered have particle excitations, but these don't interact with each other. Here we'll consider more complicated theories that include interaction terms, in the form of higher order terms in the Lagrangian.

Examples:

- **ϕ^4 theory:**
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{2.66}$$
with $\lambda \ll 1$. (This Lagrangian describes a theory in which particle number is not conserved.)

- **Scalar Yukawa Theory:**
$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - M^2\psi^*\psi - \frac{1}{2}m^2\phi^2 - g\psi^*\psi\phi \tag{2.67}$$
with $g \ll M, m$. (This theory couples a complex scalar field ψ -so not particularly relevant for us- to a real scalar ϕ: only the number of ψ particles minus the number of ψ antiparticles (aka $\bar{\psi}$) is conserved.)

2.3.2 The Interaction Picture

For a Hamiltonian $H = H_0 + H_I$, we know from Part II PQM [12] that the states in the interaction picture evolve according to:

$$i\frac{\partial\,|\phi\rangle_I}{\partial t} = H_I(t)\,|\phi\rangle_I \tag{2.68}$$

Dyson's Formula: The time evolution operator $U(t,t_0)$ evolves according to

$$i\frac{\partial U}{\partial t} = H_I(t)U \tag{2.69}$$

which is solved by *Dyson's Formula*

$$\boxed{U(t,t_0) = T\exp\left\{-i\int_{t_0}^{t} H_I(t')dt'\right\}} \tag{2.70}$$

where T stands for *time ordering* meaning that operators evaluated at later time are placed to the left.

This formula is rather formal, and we almost use its first order Taylor expansion when H_I is "small enough". In other words, time-dependent perturbation theory:

$$U(t,t_0) = I - i\int_{t_0}^{t} H_I(t')dt' + (-i)^2\int_{t_0}^{t}\int_{t_0}^{t'} dt'dt''H_I(t')H_I(t'') + \dots \tag{2.71}$$

2.3.3 Introduction to Scattering and the S-matrix

We want to compute the amplitude of transition to state $|i\rangle$ to state $|f\rangle$.
We define the *S matrix* to be

$$S \equiv U(-\infty,\infty) \tag{2.72}$$

So in particular we're interested in

$$\langle f|\, S\, |i\rangle \tag{2.73}$$

(Actually, in $\langle f|\, S - I\, |i\rangle$.)

2.3.4 Wick's Theorem

Contractions and Recovering the Propagator: It can be shown that:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : +\Delta_F(x-y) \tag{2.74}$$

$$T\psi(x)\psi^\dagger(y) =: \psi(x)\psi^\dagger(y) : +\Delta_F(x-y) \tag{2.75}$$

-We define the *contraction* of a pair of fields operators in a string of operators $\dots\phi(x_1)\dots\phi(x_2)\dots$ to be the same string except for replacing such two operators by its Feymann propagator (leaving all other operators untouched). We use the notation

$$\ldots \overbrace{\phi(x_1) \ldots \phi(x_2)} \ldots$$

Examples:

- $\overbrace{\phi(x)\phi(y)} = \Delta_F(x-y)$
- $\overbrace{\phi(x)\phi(y)\phi(z)} = \Delta_F(x-z)\phi(y)$
- $\overbrace{\phi(x)\phi(y)\phi(z)\phi(w)} = \Delta_F(x-z)\phi(y)\phi(w)$
- $\overbrace{\phi(x)\,\overbrace{\phi(y)\phi(z)}\,\phi(w)} = \Delta_F(x-w)\Delta_F(y-z)$

Recall that the Feynman propagator is a c-number, so it doesn't matter its order in the string.
For complex fields, we define similarly the contraction:

$$\overbrace{\psi(x)\psi^\dagger(y)} = \Delta_F(x-y) \tag{2.76}$$

$$\overbrace{\psi^\dagger(x)\psi^\dagger(y)} = 0 = \overbrace{\psi(x)\psi(y)} \tag{2.77}$$

Wick's Theorem For any contraction of fields $\phi_i \equiv \phi(x_i)$, we have

$$\boxed{T(\phi_1 \ldots \phi_n) =: \phi_1 \ldots \phi_n : + : \text{all possible contractions} :} \tag{2.78}$$

Scattering Wick's theorem is very useful when computing scattering amplitudes. It allows us to go from time ordered strings of fields (which is the form we have for U) to normal ordered strings. Normal ordered strings allows us to simply the integrals we get, because annihilation operators will destroy many of the terms (particles) they will encounter at its left.

2.3.5 Feynman Diagrams

Feynman diagrams can simplify notably the computations. We will, however, not use them. For completeness I'll include them in so far as it is relevant for ϕ^4 Theory. (Which is a simpler case).

The magical insight is that *every* term given by Wick's theorem can be interpreted as a diagram of this sort. Moreover, the term contributes to the process if and only if it has the right "incoming" and "outgoing" particles. So we can figure out what terms contribute by drawing the right diagrams.

Moreover, not only can we count the diagrams. We can also read out how much each term contributes from the diagram directly! This simplifies the computation a lot.

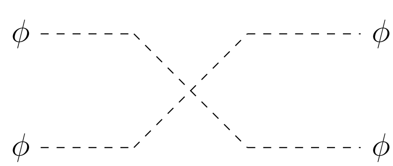
We will not provide a proof that Feynman diagrams do indeed work, as it would be purely technical and would also involve the difficult work of what it actually means to be a diagram.

We begin by specifying what diagrams are "allowed", and then specify how we assign numbers to diagrams.

Given an initial state and final state, the possible Feynman diagrams are specified as follows: Suppose we are given an initial state $|i\rangle$ and final state $|f\rangle$.
In the following example, we'll work with $|i\rangle = |\vec{p_1}, \vec{p_2}\rangle$ and $|f\rangle = \left|\vec{p_1}', \vec{p_2}'\right\rangle$. So we are considering the scattering process $\phi + \phi \to \phi + \phi$ (meson scattering).
In the ϕ^4 Theory, a *Feynman diagram* consists of:

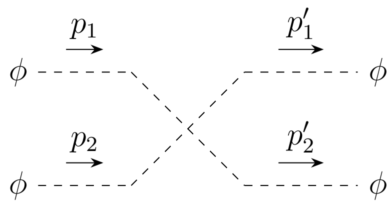
- An external line for all particles in the initial and final states.

- We join the lines together with more lines and vertices so that the only loose ends are the initial and final states. The possible vertices correspond to the interaction terms in the Lagrangian. For example, the only interaction term in the Lagrangian in this theory is ϕ^4, so the only possible vertex is one that joins four lines.



Each such vertex represents an interaction.

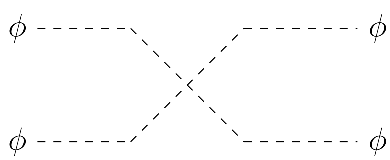
- Assign a directed momentum p to each line, i.e. an arrow into or out of the diagram to each line.



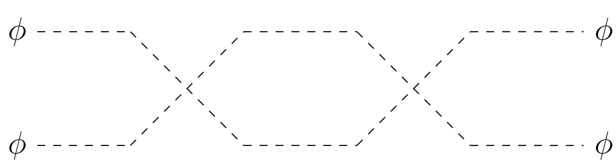
The initial and final particles already have momentum specified in the initial and final state, and the internal lines (if any -none here!) are given "dummy" momenta k_i (which we will later integrate over).

Note that there are infinitely many possible diagrams! However, when we lay down the Feynman rules later, we will see that the more vertices the diagram has, the less it contributes to the sum. In fact, the n-vertices diagrams correspond to the nth order term in the expansion of the S-matrix. So most of the time it suffices to consider "simple" diagrams.

-Example: If we look at $\phi + \phi \to \phi + \phi$, the simplest diagram is what we've drawn before. (Or actually, a diagram with no vertex where nothing happens.)



On the other hand, we can have two vertices:



This, for example, corresponds to second-order terms.

There are also more complicated ones such as things we loops. (If we ignore the loops, we say we are looking at the *tree level*.)

2.3.6 Amplitudes

We define the amplitude $\mathcal{A}_{fi}$ by

$$\langle f| S - I |i\rangle = i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(p_F - p_I) \tag{2.79}$$

where p_I and p_F are the sum of the initial and final 4-momenta, respectively.

Amplitudes in ϕ^4 Theory The Feynman rules to compute the amplitude iA_{fi} are as follows:

- Draw all possible diagrams with appropriate external legs and impose 4-momentum conservation at each vertex.
- Write down a factor of $(-i\lambda)$ at each vertex.
- For each internal line, write down the propagator.
- Integrate over momentum k flowing through each loop $\int \frac{d^4k}{(2\pi)^4}$

Using these rules, the scattering amplitude for $\phi\phi \to \phi\phi$ (drawn before) is simply $i\mathcal{A} = -i\lambda$

2.3.7 Correlation functions and vacuum bubbles

Previously, we have been working with the vacuum of the free theory $|0\rangle$. This satisfies the boring relation

$$H_0 |0\rangle = 0.$$

However, when we introduce an interaction term, this is no longer the vacuum. Instead we have an interacting vacuum $|\Omega\rangle$, satisfying

$$H |\Omega\rangle = 0.$$

As before, we normalize the vacuum so that

$$\langle\Omega|\Omega\rangle = 1.$$

Concretely, this interaction vacuum can be obtained by starting with a free vacuum and then letting it evolve for infinite time. Now given that we have

this vacuum, we can define the *correlation function*.
The *correlation* or *Green's function* is defined as

$$G^{(n)}(x_1, \cdots, x_n) = \langle\Omega| T\phi_H(x_1)\cdots\phi_H(x_n)|\Omega\rangle,$$

where ϕ_H denotes the operators in the Heisenberg picture.

How can we compute these things? It can be shown that:

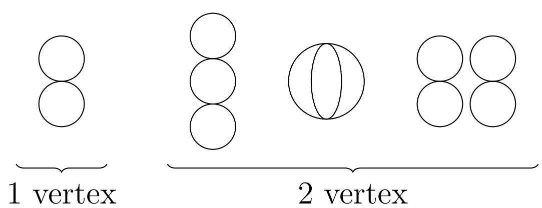
$$\boxed{G^{(n)}(x_1, \cdots, x_n) = \frac{\langle 0| T\phi_I(x_1)\cdots\phi_I(x_n)S|0\rangle}{\langle 0| S|0\rangle}}.$$

where the ϕ_I denotes the operators in the Interaction picture..

Now what does this quantity tell us? It turns out these have some really nice physical interpretation. Let's look at the terms $\langle 0| T\phi_I(x_1)\cdots\phi_I(x_n)S|0\rangle$ and $\langle 0| S|0\rangle$ individually and see what they tell us.

For simplicity, we will work with the ϕ^4 theory, so that we only have a single ϕ field, and we will, without risk of confusion, draw all diagrams with solid lines.

Looking at $\langle 0| S|0\rangle$, we are looking at all transitions from $|0\rangle$ to $|0\rangle$. The Feynman diagrams corresponding to these would look like



These are known as vacuum bubbles. Then $\langle 0| S|0\rangle$ is the sum of the amplitudes of all these vacuum bubbles.

While this sounds complicated, a miracle occurs. It happens that the different combinatoric factors piece together nicely so that we have

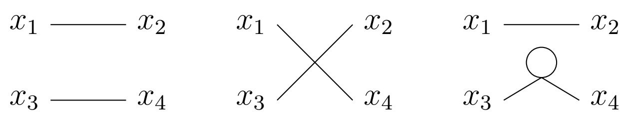
$$\langle 0| S|0\rangle = \exp\{\text{all distinct (connected) vacuum bubbles}\}.$$

Similarly, magic tells us that

$$\langle 0| T\phi_I(x_1)\cdots\phi_I(x_n)S|0\rangle = \left(\sum \begin{matrix}\text{connected diagrams}\\ \text{with } n \text{ loose ends}\end{matrix}\right)\langle 0| S|0\rangle.$$

So what $G^{(n)}(x_1, \cdots, x_n)$ really tells us is the sum of connected diagrams modulo these silly vacuum bubbles.

The diagrams that correspond to $G^{(4)}(x_1, \cdots, x_4)$ include things that look like



Note that we define "connected" to mean every line is connected to some of the end points in some way, rather than everything being connected to everything.

We can come up with analogous Feynman rules to figure out the contribution of all of these terms.

3 Free QFT in FLRW spacetime. Primordial power spectrum from inflation

In this section, we are back the mostly plus signature $(-,+,+,+)$, as opposed to the previous section (!).
This section is heavely based on [17].
We would like to quantize the inflationary model discussed in section 1.4. In particular, we will work with perturbations around the homogenous -space independent- background (now denoted with a bar) and promote them to quantum operators:

$$g_{\mu\nu}(t,\vec{x}) = \bar{g}_{\mu\nu}(t) + \hat{h}_{\mu\nu}(t,\vec{x}), \qquad \phi(t,\vec{x}) = \bar{\phi}(t) + \hat{\varphi}(t,\vec{x}) \tag{3.1}$$

Where the hat means that these are quantum operators (since fourier transforms will only be distinguised by the argument $\vec{k}$ We will first deal with perturbations of the scalar field.

3.1 Massless scalar in de Sitter

We start with a massless scalar field in de Sitter spacetime without any classical background $\bar{\phi}(t) = 0$. (Which turns out to be a good approximation of inflationary models.)
We use the action

$$S = -\int d^4x\sqrt{-g}\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi = \int d^3x dt a^3 \frac{1}{2}\left(\dot{\varphi}^2 - \frac{1}{a^2}\|\nabla\varphi\|^2\right) \tag{3.2}$$

Similarly to the case we saw in the inflation section, where we obtained (1.54), one finds that the equation of motion for φ is

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\nabla^2\varphi}{a^2} = 0 \tag{3.3}$$

In analogy to (2.24) we can write the solution in fourier space, as a quantum operator:

$$\varphi(\mathbf{k},t) = f_k(t)a_{\mathbf{k}} + f_k^*(t)a^\dagger_{-\mathbf{k}} \tag{3.4}$$

where the creation and annihilation operators obey (2.27).
Note that we are working in the Heisenberg picture, because φ as written

above depends on time.
The EoM (3.3) implies that the *mode functions* $f_k(t)$ obey

$$(af_k)'' + \left(k^2 - \frac{a''}{a}\right)(af_k) = 0 \tag{3.5}$$

where we are using conformal time ($' \equiv \partial/\partial\tau$). In de Sitter, $a''/a = 2/\tau^2$. One can check that the general solution is given by

$$f_k = \alpha(1 + ik\tau)e^{-ik\tau} + \beta(1 - ik\tau)e^{+ik\tau} \tag{3.6}$$

for integration constants α, β.
In the far past, i.e. $k\tau \gg 1$, equation (3.3) reduces to that of a simple harmonic oscillator. So the mode $a\varphi(\mathbf{k})$ is effectively in Minkowski spacetime. In this limit, we should recover the for a free scalar field in the Heisenberg. picture (2.49), using the physical wavevector ($\mathbf{k}_{phys} = \mathbf{k}/a$).
So to determine the integration constants, we match the solution for φ and its time-derivative to the Minkowski solution from free QFT in Minkowski (2.49). (Note that we have changed the metric convention for this chapter!). If the matching is done in the infinite past ($k\tau_* \to \infty$), it can be shown (see [17]) that

$$|\alpha| = \frac{H}{\sqrt{2k^3}} \qquad\qquad \beta = 0 \tag{3.7}$$

So that

$$\boxed{f_k = \frac{H}{\sqrt{2k^3}}(1 + ik\tau)e^{-ik\tau}} \tag{3.8}$$

Note that the conjugate momentum is given by

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = a^3\dot{\varphi} \tag{3.9}$$

The *two-point correlator* of φ is

$$\lim_{\tau\to 0}\langle\varphi(\mathbf{k})\varphi(\mathbf{k}')\rangle = \lim_{\tau\to 0} f_k f^*_{-k}\langle a_{\mathbf{k}} a^\dagger_{-\mathbf{k}}\rangle \equiv (2\pi)^3\delta^{(3)}(\mathbf{k} + \mathbf{k}')P(k) \tag{3.10}$$

where we have introduced the *power spectrum*

$$\boxed{P(k) = \frac{H^2}{2k^3} = \lim_{\tau\to 0}|f_k|^2} \tag{3.11}$$

after using (3.8).
The limit $\tau \to 0$ corresponds to the infinite future of de Sitter space. Because we only have observational access to what happened after inflation. And we are modelling the early universe as de Sitter, which of course is not, otherwise it would last forever. But under this assumption, the end of inflation is at the infinite future, which for Minkowski corresponds to $\tau \propto 1/a \to \infty$.
The k-dependence $P \propto k^{-3}$ corresponds to *scale invariance*, which we saw in 1.3.3. If we Fourier Transform, the real-space *correlation function* does not depend on distance:

$$\langle \varphi(\mathbf{x})\varphi(\mathbf{0})\rangle = \int \frac{d^3k'}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\langle \varphi(\mathbf{k})\varphi(\mathbf{k}')\rangle \sim H^2 \tag{3.12}$$

which doesn't change if we rescale $\mathbf{x} \to \lambda\mathbf{x}$.
In Minkowski, instead, we have $P(k) = 1/2k$. We'll show why:

$$\lim_{\tau\to 0}\langle \varphi(\mathbf{k})\varphi(\mathbf{k}')\rangle = \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}\langle (a_{\mathbf{k}} + a^\dagger_{-\mathbf{k}})(a_{\mathbf{k}'} + a^\dagger_{-\mathbf{k}'})\rangle \tag{3.13}$$

$$= \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}\langle 0| a_{\mathbf{k}} a^\dagger_{-\mathbf{k}'} |0\rangle \tag{3.14}$$

$$= \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{-\mathbf{k}}}}(2\pi)^3\delta^{(3)}(\mathbf{k}' + \mathbf{k}) \tag{3.15}$$

$$= \frac{1}{2k}(2\pi)^3\delta^{(3)}(\mathbf{k}' + \mathbf{k}) \tag{3.16}$$

where in the first line, we have used (2.24). So in Minkowski,

$$\langle \varphi(\mathbf{x})\varphi(\mathbf{0})\rangle \sim \frac{1}{x^2}$$

which is <u>not</u> scale-invariant.

3.2 Massive scalar in de Sitter

We can also consider a field with a mass, so that the action is now

$$S = -\int \sqrt{-g}\frac{1}{2}[\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2] \tag{3.17}$$

The field φ is related to creation and annihilation operators as before, via (add ref). But the mode functions are now modified

$$f_k(\tau) = \frac{\sqrt{\pi}H}{2}(-\tau)^{3/2}H^{(1)}_\nu(-k\tau) \tag{3.18}$$

where

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$$

and $H_\nu^{(1)} \equiv J_\nu + iY_\nu$ is a Hankel function of the first kind [6]. Taylor-expanding around $\tau \to 0$, there two different cases:

1. $m^2 < 9H^2/4$: This implies $P(k) \propto (-k\tau)^{3-2\nu}/k^3$ so it is not scale invariant anymore. P has now a time dependence, and for $m^2 > 0$, it decays with time and vanishes at infinity.

2. $m^2 > 9H^2/4$: Here ν is complex. The power spectrum oscillates while decaying as τ^3.

In cosmology, we are mostly interested in massless or almost massless fields, which do not create large instabilities and whose perturbations survive long enough to be observable at late times.

3.3 Gravitons in de Sitter

We now will quantize quantum metric fluctuations:

$$g_{\mu\nu}(\mathbf{x}, t) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(\mathbf{x}, t)$$

By means of symmetry, the contracted Bianchi identity (1.2), and a suitable gauge transformation, we can be left with two dynamical components. These describe the two helicities of the graviton, $h = \pm 2$. A convenient gauge choice to study the linear dynamics of gravitons on a FLRW spacetime is

$$ds^2 = -dt^2 + a^2(\delta_{ij} + \gamma_{ij})dx^i dx^j$$

where $\partial_i \gamma_{ij} = 0 = \gamma_{ij}$. This is the transverse traceless gauge [7] . Expanding the Einstein-Hilbert action (1.10) to quadratic order in γ_{ij}, one gets

$$S_2 = \frac{M_{Pl}^2}{8} \int d^3x d\tau a^2 [\gamma'_{ij}\gamma'_{ij} - \partial_k \gamma_{ij} \partial_k \gamma_{ij}] \tag{3.19}$$

We expand the gravition in plane waves by writing

$$\gamma_{ij}(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \sum_{s=+,\times} \epsilon_{ij}^s(\mathbf{k}) \gamma_s(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{3.20}$$

[6] https://mathworld.wolfram.com/HankelFunctionoftheFirstKind.html

[7] We saw something very similar in Part II GR [2] when studying gravitational waves.

where $\epsilon^s_{ij}(\mathbf{k})$ are *polarization tensors*. For their expression, see [17]. We can then rewrite the action (3.19) so that S_2 consists now of two independent copies of the action for a massless scalar field (3.2). The two polarizations $\gamma_{+,\times}$ are now canonically normalized. To quantize the theory, we write, as before,

$$\gamma_s(\mathbf{k}) = \left(f_k a^s_{\mathbf{k}} + f^*_k a^{s\dagger}_{-\mathbf{k}}\right) \frac{M_{pl}}{\sqrt{2}} \tag{3.21}$$

with the usual commutation relations (2.27) for both $a^+, a^\times$ (whose commutator vanishes, i.e.: $[a^\times, a^+] = [(a^\times)^\dagger, a^+] = 0$).
If we assume a dS background, i.e. $a = e^{Ht}$, the mode functions f_k are the same as for a massless scalar field (3.6). The gravitation power spectrum, often called the *tensor power spectrum* $P_T(k)$ is given by

$$\lim_{\tau\to 0}\langle\gamma_{ij}(\mathbf{k})\gamma_{ij}(\mathbf{k}')\rangle \equiv (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')P_T(k)$$

This implies that

$$P_T = \frac{4H^2}{k^3 M^2_{Pl}}$$

We have again used (3.8).

4 Interacting QFT in Cosmology

This section follows [17] very closely. It will use the main results from section 2.3 on interacting QFT. Naturally, it also uses results from free QFT in Cosmology.

4.1 The in-in formalism: cosmology and correlators

In particle physics, we were interested in the S-matrix arising from the scattering amplitudes (see 2.3.6) for an state $|\text{in}, \alpha\rangle$ to evolve to another state $|\text{out}, \beta\rangle$. In cosmology, the situation is different:

In-in vs in-out: At early times, cosmological perturbations were effectively in flat space and we can define an initial state (just as in 2.3). However, at late times, cosmological perturbations evolve and interact with each other. We cannot assume that the state of the universe at late times is a superposition of free states (as we did for particle scattering). So instead of "in-out" amplitudes, we care about"in-in" expectation values.
For an operator $\mathcal{O}$, we define its *in-in correlator* as

$$\langle \mathcal{O} \rangle = \langle \Omega | \, \mathcal{O} \, | \Omega \rangle$$

we will take $|\Omega\rangle$ to be the "vacuuum" of the interactig theory (so $\lim_{\tau \to -\infty} |\Omega\rangle = |0\rangle$), and $\mathcal{O}$ will always be the <u>equal-time</u> product of operators at different space points. (So time ordering is irrelevant).
Notes:

- Correlators of Hermitian operators are <u>observables</u> and, unlike scattering amplitudes, must be real (easy proof).
- Explicit calculations are most easily performed in the interaction picture (section 2.3.2).

It can be shown that, under a suitable approximation,

$$\boxed{\langle \mathcal{O}(\tau) \rangle = \langle 0 | \left[\bar{T} e^{i \int_{-\infty(1+i\epsilon)}^{\tau} d\tau' H_{int}(\tau')} \right] \mathcal{O}(\tau) \left[T e^{-i \int_{-\infty(1-i\epsilon)}^{\tau} d\tau' H_{int}(\tau')} \right] |0\rangle} \tag{4.1}$$

an equivalent version of this formula is

$$\langle\mathcal{O}(\tau)\rangle = \sum_{N=0}^{\infty} i^N \int_{-\infty}^{\tau} d\tau_N \int_{-\infty}^{\tau_N} d\tau_{N-1} \cdots \int_{-\infty}^{\tau_2} d\tau_1 \\ \times \langle 0| [H_{int}(\tau_1), [H_{int}(\tau_2), [\ldots [H_{int}(\tau_N), \mathcal{O}(\tau)] \ldots]]] |0\rangle$$

These are the *factorized form* and the *commutator form*.
In perturbation theory, the first few terms are:

$$\langle 0| \mathcal{O}(\tau) |0\rangle \tag{4.2}$$

to zero-th order, and

$$i \int_{-\infty}^{\tau} \langle 0| [H_{int}(\tau'), \mathcal{O}(\tau)] |0\rangle \tag{4.3}$$

to first order. We also need a version of Wick's theorem (2.78): Using the notation $\varphi_a = \varphi(\mathbf{k}_a, \tau_a)$, we have

$$\varphi_1 \ldots \varphi_{2n} = \sum_{\text{all possible pairwise contractions}} : \varphi_1 \ldots \varphi_{2n} : \tag{4.4}$$

Since $\langle : \mathcal{O} : \rangle = 0$, inside an expectation value, the only surviving term is that in which all fields have been contracted,

$$\boxed{\langle \varphi_1 \ldots \varphi_{2n} \rangle = \sum_{\text{perms}} \langle \varphi_1 \varphi_2 \rangle \ldots \langle \varphi_{2n-1} \varphi_{2n} \rangle} \tag{4.5}$$

In what follows, we will need the following results:

- For an Hermitian operator $\mathcal{O}$,

$$\langle [H_{int}, \mathcal{O}] \rangle = 2i \mathrm{Im} \langle H_{int} \mathcal{O} \rangle \tag{4.6}$$

- We will assume that the theory is symmetric under spatial parity $\Longrightarrow$ the theory is symmetric under momentum parity $\Longrightarrow$ the product of equal-time fields in Fourier space is an Hermitian operator.
  In particular, (4.6) applies when $\mathcal{O}$ is a product if equal-time fields in Fourier space.

4.1.1 Examples of contact correlators

We will apply the above formalism to some examples. These will be *contact interaction*, which contribute to correlators already at linear order in H_{int}.

Cubic interactions: The simplest interaction one can think of in particle physics is a cubic potential term $V = \mu\varphi(x)^3$. We can write the corresponding Hamiltonian as

$$H_{int}(\tau) = -\mathcal{L}_{int} = V = \int d^3x\sqrt{-g}\mu\varphi(\mathbf{x},\tau)^3 \tag{4.7}$$

$$= a^4\mu\int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}\int d^3x e^{i\mathbf{x}\cdot(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3)}\varphi(\mathbf{q}_1,\tau)\varphi(\mathbf{q}_2,\tau)\varphi(\mathbf{q}_3,\tau) \tag{4.8}$$

$$= a^4\mu\int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}\varphi(\mathbf{q}_1,\tau)\varphi(\mathbf{q}_2,\tau)\varphi(\mathbf{q}_3,\tau)(2\pi)^3\delta^{(3)}(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3) \tag{4.9}$$

where $\sqrt{-g} = a^4$ (instead of a^3) appears because we are using conformal time τ,

$$\int_{\mathbf{q}} \equiv \int \frac{d^3q}{(2\pi)^3}$$

and φ is in (3.4) with the mode functions also as before (3.6). This interaction induces a non-vanishing *three-point correlator* or *bispectrum*. Its leading non-trivial order is given by (4.3):

$$\langle\varphi(\mathbf{k}_1,\tau)\varphi(\mathbf{k}_2,\tau)\varphi(\mathbf{k}_3,\tau)\rangle = i\int_{-\infty}^{\tau} d\tau'\langle[H_{int}(\tau'),\varphi(\mathbf{k}_1,\tau)\varphi(\mathbf{k}_2,\tau)\varphi(\mathbf{k}_3,\tau)]\rangle$$
$$= -2\mu\mathrm{Im}\int_{-\infty}^{\tau} d\tau' a^4(\tau')\int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}\langle\varphi(\mathbf{q}_1,\tau')\varphi(\mathbf{q}_2,\tau')\varphi(\mathbf{q}_3,\tau')$$
$$\times\varphi(\mathbf{k}_1,\tau)\varphi(\mathbf{k}_2,\tau)\varphi(\mathbf{k}_3,\tau)\rangle(2\pi)^3\delta^{(3)}(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3)$$

where we have used (4.6) to obtain the last line. Using $a(\tau) = 1/H\tau$ (for de Sitter) and Wick's theorem (4.5), noting that the delta function kills equal-time correlators (unless $\mathbf{k}_i = 0$ for $i = 1, 2, 3$) and exploiting the symmetry over the $\mathbf{k}_i$'s, we get

$$\begin{aligned}
\langle\varphi(\mathbf{k}_1,\tau)\varphi(\mathbf{k}_2,\tau)\varphi(\mathbf{k}_3,\tau)\rangle &= -2\mu\times 3!\times \mathrm{Im}\int_{-\infty}^{\tau}\frac{d\tau'}{(H\tau')^4}\int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}(2\pi)^3 \\
&\quad\times\delta^{(3)}(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3)\prod_{a=1}^{3}\langle\varphi(\mathbf{q}_a,\tau')\varphi(\mathbf{k}_a,\tau)\rangle \\
&= -2\mu\times 3!\times \mathrm{Im}\int_{-\infty}^{\tau}\frac{d\tau'}{(H\tau')^4}\int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}(2\pi)^3\delta^{(3)}(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3) \\
&\quad\times\prod_{a=1}^{3}\left[f_{q_a}(\tau')f^*_{k_a}(\tau)(2\pi)^3\delta^{(3)}(\mathbf{k}_a+\mathbf{q}_a)\right] \\
&= -2\mu\times 3!\times(2\pi)^3\delta^{(3)}(-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \\
&\quad\times\mathrm{Im}\prod_{a=1}^{3}\left[f^*_{k_a}(\tau)\right]\int_{-\infty}^{\tau}\frac{d\tau'}{(H\tau')^4}\prod_{b=1}^{3}\left[f_{q_b}(\tau')\right] \\
&= -2\mu\times 3!\times(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\mathrm{Im}\prod_{a=1}^{3}\left[f^*_{k_a}(\tau)\right]\int_{-\infty}^{\tau}\frac{d\tau'}{(H\tau')^4}\prod_{b=1}^{3}\left[f_{q_b}(\tau')\right]
\end{aligned}$$

We recognize the momentum-conserving delta functions. It is common to suppress this factor by appending a prime to the correlator or define B_n as

$$\langle\varphi(\mathbf{k}_1)\dots\varphi(\mathbf{k}_n)\rangle \equiv (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\dots+\mathbf{k}_n)B_n(\mathbf{k}_1,\dots,\mathbf{k}_n) \tag{4.10}$$

$$\langle\varphi(\mathbf{k}_1)\dots\varphi(\mathbf{k}_n)\rangle' \equiv B_n(\mathbf{k}_1,\dots,\mathbf{k}_n) \tag{4.11}$$

Using the expression for the mode functions (3.6), we get that B_3 is equal to

$$-\frac{3}{2}\frac{\mu H^2}{(k_1k_2k_3)^3}\mathrm{Im}\left(\left[\prod_{a=1}^{3}(1-ik_a\tau)\right]\int_{-\infty}^{\tau}\frac{d\tau'}{(\tau')^4}e^{-ik_T(\tau'-\tau)}\left[\prod_{b=1}^{3}(1+ik_b\tau')\right]\right) \tag{4.12}$$

Where we have defined the "total momentum" $k_T = k_1+k_2+k_3$. After expanding the product in the integrand, one finds integrals of the form

$$I_n = \int_{-\infty}^{\tau}\frac{d\tau'}{(\tau')^n}e^{-ik_T\tau'}$$

for $n = 1,2,3,4$. Integrating by parts, one has, for $n>1$:

$$I_n = -\frac{1}{n-1}\frac{1}{\tau^{n-1}}e^{-ik_T\tau}+\frac{(-ik_T)}{n-1}I_{n-1}$$

We want to reduce each term to I_1, and from it, to the exponential integral

$$\text{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt \tag{4.13}$$

We can relate I_1 and Ei as follows:

$$I_1 = \int_{-\infty}^{\tau} \frac{e^{-ik_T\tau'}}{\tau'} d\tau' = -\int_{\tau}^{\infty} \frac{e^{-ik_T\tau'}}{\tau'} d\tau' = \text{Ei}(-ik_T\tau)$$

The first equality follows from Jordan's lemma (closing the contour). The second one follows from a change of variables (closing again at infinity by Jordan's lemma).

We are interested in the limit of $\tau \to 0$ of B_3. After a lot of algebra, some cancellations with the other product in (4.12), using the (divergent!) expansion [18]

$$\text{Ei}(-ik_T\tau) \sim \gamma_E + \log k_T\tau - i\frac{\pi}{2} \tag{4.14}$$

and taking the imaginary part as required, one gets

$$\boxed{B_{\varphi^3} \sim \frac{\mu H^2}{2(k_1k_2k_3)^3} \left[\sum_a k_a^3(\gamma_E - 1 + \log|k_T\tau|) + k_1k_2k_3 - \sum_{a \neq b} k_a^2 k_b \right]} \tag{4.15}$$

as $\tau \to 0^-$, where here γ_E is the Euler-Mascheroni constant [19]

Quartic interaction Suppose now that the contribution comes from the contact interaction

$$H_{int} = \int d^3x \frac{a^4}{4!\lambda^4} \dot{\varphi}^4(\mathbf{x}, \tau) \tag{4.16}$$

$$= \int_{\mathbf{q}_1, \dots \mathbf{q}_4} \frac{(2\pi)^3}{4!\lambda^4} \varphi'(\mathbf{q}_1, \tau) \dots \varphi'(\mathbf{q}_4, \tau) \delta^{(3)}(\mathbf{q}_1 + \dots + \mathbf{q}_4) \tag{4.17}$$

$$\implies B_4 = -\frac{2 \times 4!}{4!\lambda^4} \text{Im} \left(\left[\prod_{a=1}^{4} f_{k_a}(\tau) \right] \int_{-\infty}^{\tau} d\tau' \left[\prod_{a=1}^{4} f_{q_a}(\tau') \right] \right) \tag{4.18}$$

Here dot, $\dot{}$, denotes a time derivative t, while ' denotes a derivative with respect to the conformal time .The final step uses again the trick (4.6) and Wick's theorem (4.5). The derivative of the mode function is

$$f_k'(\tau) = \frac{H}{\sqrt{2k^3}} k^2 \tau e^{-ik\tau} \tag{4.19}$$

so we will need the result[8]

$$\int_{-\infty}^{0} d\tau' e^{-ik_T\tau'}(\tau')^p = -\frac{(-i)^{p+1}p!}{k_T^{p+1}} \tag{4.20}$$

for $p = 4$. Taking the limit $\tau \to 0$, one gets

$$\boxed{B_{\dot\varphi^4} \sim -\frac{3H^8}{\lambda^4}\frac{1}{k_T^5 k_1 k_2 k_3 k_4}} \tag{4.21}$$

4.2 Computing correlators from inflation ($P(X,\phi)$ theories)

4.2.1 $P(X,\phi)$ at quadratic order and the speed of sound

Assuming we have found some solution to the background equation of motion (1.63) of some $P(X,\phi)$ theory (1.4.2), we allow for perturbations

$$\phi(\mathbf{x},t) = \bar\phi(t) + \varphi(\mathbf{x},t) \tag{4.22}$$

with $|\varphi| \ll |\bar\phi|$. We write

$$\delta X = X - \bar X \tag{4.23}$$

where $\bar X = -\frac{1}{2}\partial_\mu\bar\phi\partial^\mu\bar\phi$.
The Lagrangian can be expanded in φ.
Its part linear on φ is

$$\mathcal{L}_1 = P_\phi\varphi + P_X\dot{\bar\phi}\dot\varphi \tag{4.24}$$

and so the corresponding Euler-Lagrange equations are

$$\frac{\partial\mathcal{L}_1}{\partial\varphi} = \frac{d}{dt}\frac{\partial\mathcal{L}_1}{\partial\dot\varphi} \tag{4.25}$$

which can be easily check to be equivalent to the equation of motion for the background $\bar\phi(t)$, (1.63), which we assumed to be satisfied.
Focusing on the quadratic term, and integrating by parts the quadratic Lagrangian, one gets the quadratic action

$$S_2 = \int d^3x dt a^3 \frac{1}{2}\left[(P_X + 2P_{XX}\bar X)\dot\varphi^2 - P_X\partial_i\varphi\partial^i\varphi - m^2\varphi\right] \tag{4.26}$$

$$m^2 \equiv 3HP_{X\phi}\dot{\bar\phi} + \partial_t\left(P_{X\phi}\dot{\bar\phi}\right) - P_{\phi\phi} \tag{4.27}$$

[8]It can be proven by a change of variables and contour integration, to relate the integral to the gamma function.

Here P and its derivatives are evaluated on the background $P = P(\bar{X}, \bar{\phi})$. It can be shown that the mass term (4.27) is negligible compared to the others [17], because all background quantities involving derivatives with respect to ϕ are suppressed by slow-roll parameters.
Droping hence the mass term and rescaling $\varphi \mapsto \sqrt{P_X}\varphi$, the quadratic action (4.26) becomes

$$S_2 \approx \int d^3xdta^3\frac{1}{2}\left[c_s^{-2}\dot{\varphi}_c^2 - \partial_i\varphi\partial^i\varphi\right] \tag{4.28}$$

$$= \int d^3xdta^3\frac{1}{2}\left[c_s^{-2}\dot{\varphi}_c^2 - a^{-2}\partial_i\varphi\partial_i\varphi\right] \tag{4.29}$$

$$c_s^2 \equiv \frac{P_X}{P_X + 2P_{XX}\bar{X}} \tag{4.30}$$

where we call c_s the *speed of sound* and treat it as constant (we assume its time dependence to be weak, due again to the slow-roll conditions).From dimensional analysis, we see that c_s must have units of $(\text{lenght})^2/(\text{time})^2$, but the right hand side of (4.30) is dimensionless, so we really mean

$$c_s^2 = c^2\frac{P_X}{P_X + 2P_{XX}\bar{X}} \tag{4.31}$$

This form of the action is familiar, it is (3.2), and leads to the also familiar equation of motion (3.3), except we had $c_s = 1$ in the previous section. The mode functions are now

$$f_k(\tau) = \frac{H}{\sqrt{2c_sk^3}}(1 + ic_sk\tau)e^{-ic_sk\tau} \tag{4.32}$$

$$\implies f_k'(\tau) = \frac{H}{\sqrt{2c_sk^3}}c_s^2k^2\tau \tag{4.33}$$

By dimensional analysis and comparing with the known result (3.11) for the power spectrum in the case $c_s = 1$, it can be shown that

$$P(k) = \frac{H^2}{2c_sk^3} \tag{4.34}$$

This can also be calculated from first principles.

4.2.2 Cubic interactions in $P(X,\phi)$ theories

The cubic order Lagrangian for a $P(X,\phi)$ is:

$$\begin{aligned}\mathcal{L}_3 =& \frac{1}{6}P_{XXX}\dot{\bar{\phi}}^3\dot{\varphi}^3 - \frac{1}{2}P_{XX}\dot{\bar{\phi}}\dot{\varphi}(\partial_\mu\varphi)(\partial^\mu\varphi) \\ &- \frac{1}{2}P_{X\phi}\varphi(\partial_\mu\varphi)(\partial^\mu\varphi) + \frac{1}{2}P_{X\phi\phi}\dot{\bar{\phi}}\dot{\varphi}\varphi^2 + \frac{1}{2}P_{XX\phi}\dot{\bar{\phi}}^2\dot{\varphi}^2\varphi + \frac{1}{6}P_{\phi\phi\phi}\varphi^3 \end{aligned} \tag{4.35}$$

$$\begin{aligned} =& \frac{1}{6}P_{XXX}\dot{\bar{\phi}}^3\dot{\varphi}^3 + \frac{1}{2}P_{XX}\dot{\bar{\phi}}\dot{\varphi}^3 - \frac{1}{2}P_{XX}\dot{\bar{\phi}}\dot{\varphi}(\partial_i\varphi)(\partial_i\varphi)a^{-2} + \\ &- \frac{1}{2}P_{X\phi}\varphi(\partial_i\varphi)(\partial_i\varphi)a^{-2} + \frac{1}{2}P_{X\phi}\dot{\bar{\phi}}\dot{\varphi}\varphi^2 + \frac{1}{2}(P_{XX\phi}\dot{\bar{\phi}} + P_{X\phi})\dot{\varphi}^2\varphi + \frac{1}{6}P_{\phi\phi\phi}\varphi^3 \end{aligned} \tag{4.36}$$

after using $g^{\mu\nu} = \mathrm{diag}(-1, a^{-2}, a^{-2}, a^{-2})$ and $\partial_\mu\varphi\partial^\mu\varphi = g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi)$.
Note: It can be shown (see [17]) that the terms with the ∂_ϕ derivative are suppressed (by slow-roll).
We write

$$\mathcal{L}_{(\varphi')^3} = \frac{1}{6}A\dot{\varphi}^3 \tag{4.37}$$

$$A \equiv P_{XXX}\dot{\bar{\phi}}^3 + 3P_{XX}\dot{\bar{\phi}} \tag{4.38}$$

$$\mathcal{L}_{\varphi'(\nabla\varphi)^2} = -\frac{1}{2}C\dot{\varphi}(\nabla\varphi)^2a^{-2} \tag{4.39}$$

$$C \equiv P_{XX}\dot{\bar{\phi}} \tag{4.40}$$

$$\mathcal{L}_{\varphi(\nabla\varphi)^2} = -\frac{1}{2}E\varphi(\nabla\varphi)^2a^{-2} \tag{4.41}$$

$$E \equiv P_{X\phi} \tag{4.42}$$

$$\mathcal{L}_{\dot{\varphi}\varphi^2} = -D\dot{\varphi}\varphi^2 \tag{4.43}$$

$$D \equiv -\frac{1}{2}P_{X\phi}\dot{\bar{\phi}} \tag{4.44}$$

$$\mathcal{L}_{\dot{\varphi}^2\varphi} = -\lambda\dot{\varphi}^2\varphi \tag{4.45}$$

$$\lambda \equiv -\frac{1}{2}(P_{XX\phi}\dot{\bar{\phi}} + P_{X\phi}) \tag{4.46}$$

A, C, E, D, λ are not really constants, but we will treat them as if they were constants, because their time dependence is assumed to be weak.

Bispectrum from $\dot{\varphi}^3$ We first compute the bispectrum arising from the term (4.37) in the cubic order Lagrangian.

$$H_{(\varphi')^3} = -\int d^3x\sqrt{-g}\mathcal{L}_{(\varphi')^3} = -\int d^3x\frac{1}{6}A\dot{\varphi}^3a^4(\tau) \tag{4.47}$$

$$= -\frac{A}{6}(2\pi)^3\int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}\left[\prod_{i=1}^{3}\varphi'(\mathbf{q}_i,\tau)\right]\delta^{(3)}\left(\sum_{i=1}^{3}\mathbf{q}_i\right)a(\tau) \tag{4.48}$$

$$\Longrightarrow$$

$$B_{(\varphi')^3} = -2A\times\mathrm{Im}\left(\left[\prod_{a=1}^{3}f^*_{k_a}(\tau)\right]\int_{-\infty}^{\tau}\frac{d\tau'}{H\tau'}\left[\prod_{b=1}^{3}f'_{k_b}(\tau')\right]\right) \tag{4.49}$$

$$\lim_{\tau\to 0}B_{(\varphi')^3} = -2A\frac{H^3}{(\sqrt{2c_s})^3}\frac{(k_1k_2k_3)^{1/2}}{(k_1k_2k_3)^{3/2}}\frac{H^3}{(\sqrt{2})^3}(c_s^{3/2})^3\mathrm{Im}\int_{-\infty}^{0}\frac{d\tau'}{H\tau'}e^{-ic_sk_T\tau'}(\tau')^3 \tag{4.50}$$

$$= \frac{-AH^5}{4}\frac{c_s^3}{k_1k_2k_3}\mathrm{Im}\left(-\frac{(-i)^3 2!}{c_s^3k_T^3}\right) \tag{4.51}$$

So the correlator is

$$\boxed{\lim_{\tau\to 0}B_{(\varphi')^3} = \frac{H^5}{2}\frac{P_{XXX}\dot{\bar{\phi}}^3 + 3P_{XX}\dot{\bar{\phi}}}{k_1k_2k_3k_T^3}} \tag{4.52}$$

Bispectrum from $\dot{\varphi}\partial_i\varphi\partial_i\varphi$ Now we will compute the correlator arising from the term (4.39):

$$\begin{aligned}
H_{\varphi'(\nabla\varphi)^2} &= -\int d^3x\sqrt{-g}\mathcal{L}_{\varphi'(\nabla\varphi)^2} = \int d^3x\frac{1}{2}C\dot{\varphi}(\nabla\varphi)^2a^2(\tau) && (4.53)\\
&= \frac{C}{2}(2\pi)^3\int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}\varphi'(\mathbf{q}_1,\tau)\varphi(\mathbf{q}_2,\tau)\varphi(\mathbf{q}_3,\tau)\\
&\times(-\mathbf{q}_2\cdot\mathbf{q}_3)\delta^{(3)}\left(\sum_{i=1}^{3}\mathbf{q}_i\right)a(\tau) && (4.54)\\
&= \frac{C}{2}(2\pi)^3\int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}\varphi'(\mathbf{q}_1,\tau)\varphi(\mathbf{q}_2,\tau)\varphi(\mathbf{q}_3,\tau)\\
&\times\frac{(-1)}{2}\left([\mathbf{q}_2+\mathbf{q}_3]^2-q_2^2-q_3^2\right)\delta^{(3)}\left(\sum_{i=1}^{3}\mathbf{q}_i\right)a(\tau) && (4.55)\\
&= \frac{C}{4}(2\pi)^3\int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}\varphi'(\mathbf{q}_1,\tau)\varphi(\mathbf{q}_2,\tau)\varphi(\mathbf{q}_3,\tau)\\
&\times\left(q_2^2+q_3^2-q_1^2\right)\delta^{(3)}\left(\sum_{i=1}^{3}\mathbf{q}_i\right)a(\tau) && (4.56)\\
&\Longrightarrow\\
B_{\varphi'(\nabla\varphi)^2} &= \frac{2C}{4}\times\mathrm{Im}\left[\prod_{a=1}^{3}f^*_{k_a}(\tau)\right]\int_{-\infty}^{\tau}\frac{d\tau'}{H\tau'}\\
&\times\sum_{\text{perms}}f'_{k_1}(\tau')f_{k_2}(\tau')f_{k_3}(\tau')\left(k_2^2+k_3^2-k_1^2\right) && (4.57)\\
\lim_{\tau\to0}B_{\varphi'(\nabla\varphi)^2} &= \frac{C}{2}\left(\frac{H}{\sqrt{2c_s(k_1k_2k_3)}}\right)^{3\times2}\mathrm{Im}\int_{-\infty}^{0}\frac{d\tau'}{H\tau'}e^{-ic_sk_T\tau'}\tau'c_s^2\\
&\times\sum_{\text{perms}}(1+ic_sk_2\tau')(1+ic_sk_3\tau')k_1^2\left(k_2^2+k_3^2-k_1^2\right) && (4.58)\\
&= \frac{C}{16}\frac{H^5}{c_s(k_1k_2k_3)^3}\sum_{\text{perms}}\mathrm{Im}F(k_1,k_2,k_3) && (4.59)
\end{aligned}$$

where

$$F(k_1,k_2,k_3) \equiv \int_{-\infty}^{0} d\tau' e^{-ic_s k_T \tau'} k_1^2 \left(k_2^2+k_3^2-k_1^2\right)\left(1+ic_s(k_2+k_3)\tau' - c_s^2 k_2 k_3 (\tau')^2\right) \tag{4.60}$$

$$= k_1^2\left(k_2^2+k_3^2-k_1^2\right)\left(\frac{i}{c_s k_T} + \frac{ic_s(k_2+k_3)}{c_s^2 k_T^2} + c_s^2 k_2 k_3 \frac{2i}{(c_s k_T)^3}\right) \tag{4.61}$$

Hence,

$$\lim_{\tau\to 0} B_{\varphi'(\nabla\varphi)^2} = \frac{C}{16}\frac{H^5}{c_s^2(k_1k_2k_3)^3 k_T^3}\sum_{\text{perms}} k_1^2\left(k_2^2+k_3^2-k_1^2\right)\left(k_T^2+k_T(k_2+k_3)+2k_2k_3\right) \tag{4.62}$$

$$= \frac{C}{8}\frac{H^5}{c_s^2(k_1k_2k_3)^3 k_T^3} S \tag{4.63}$$

where S is the following sum:

$$\begin{aligned}
S \equiv & \sum_{\text{even perms}} k_1^2\left(k_T^2 - 2k_1^2 - 2\sum_{a<b} k_a k_b\right)\left(2k_T^2 - k_1 k_T + 2k_2k_3\right)\\
=& 2k_T^4 \sum k_a^2 - k_T^3 \sum k_a^3 + 2k_1k_2k_3k_T^3 - 4k_T^2\sum k_a^4 + 2k_T \sum k_a^5\\
& - 4k_2k_3k_1 \sum k_a^3 - 4k_T^2 \sum_{a<b} k_a k_b \sum k_a^2 + 2k_T \sum k_a^3 \sum_{a<b} k_a k_b\\
& - 4k_1k_2k_3k_T \sum_{a<b} k_a k_b\\
=& 2k_T^6 - 4k_T^4 \sum_{a<b} k_a k_b - \sum k_a^3 \left(k_T^3 - 4k_1k_2k_3 - 2k_T \sum_{a<b} k_a k_b k_T\right)\\
& - 4k_T^2 \sum k_a^4 + 2k_T \sum k_a^5 + 2k_1k_2k_3k_T^3 - 4k_T^2 \sum_{a<b} k_a k_b (k_T^2 - 2\sum_{a<b} k_a k_b)\\
& - 4k_1k_2k_3k_T \sum_{a<b} k_a k_b
\end{aligned}$$

We use Mathematica to check that the sum S is equal to

$$-\frac{1}{2}\left[24(k_1k_2k_3)^2 - 8k_T(k_1k_2k_3)\left(\sum_{a<b}k_ak_b\right) - 8k_T^2\left(\sum_{a<b}k_ak_b\right)^2 \right.$$
$$\left. +22k_T^3k_1k_2k_3 - 6k_T^4\left(\sum_{a<b}k_ak_b\right) + 2k_T^6\right]$$

With this, we obtain our final expression for the correlator

$$\begin{aligned}\lim_{\tau\to 0} B_{\varphi'(\nabla\varphi)^2} =& \frac{-1}{16}\frac{P_{XX}\dot{\bar{\phi}}H^5}{c_s^2(k_1k_2k_3)^3k_T^3}\left[24(k_1k_2k_3)^2 - 8k_T(k_1k_2k_3)\left(\sum_{a<b}k_ak_b\right)\right. \\ & \left. - 8k_T^2\left(\sum_{a<b}k_ak_b\right)^2 + 22k_T^3k_1k_2k_3 - 6k_T^4\left(\sum_{a<b}k_ak_b\right) + 2k_T^6\right] \end{aligned} \tag{4.64}$$

$$\begin{aligned} =& -\frac{P_{XX}\dot{\bar{\phi}}H^5(P_X + 2P_{XX}\bar{X})}{16P_X(k_1k_2k_3)^3k_T^3}\left[24(k_1k_2k_3)^2 - 8k_T(k_1k_2k_3)\left(\sum_{a<b}k_ak_b\right)\right. \\ & \left. - 8k_T^2\left(\sum_{a<b}k_ak_b\right)^2 + 22k_T^3k_1k_2k_3 - 6k_T^4\left(\sum_{a<b}k_ak_b\right) + 2k_T^6\right] \end{aligned} \tag{4.65}$$

where the last step follows from using the expression (4.31) with $c = 1$.

Bispectrum from $\varphi\partial_i\varphi\partial_i\varphi$

$$
\begin{aligned}
H_{\varphi(\nabla\varphi)^2} &= -\int d^3x\sqrt{-g}\mathcal{L}_{\varphi(\nabla\varphi)^2} = \int d^3x\frac{1}{2}E\varphi(\nabla\varphi)^2a^2(\tau) && (4.66)\\
&= \frac{E}{2}(2\pi)^3\int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}\varphi(\mathbf{q}_1,\tau)\varphi(\mathbf{q}_2,\tau)\varphi(\mathbf{q}_3,\tau)\\
&\times(-\mathbf{q}_2\cdot\mathbf{q}_3)\delta^{(3)}\left(\sum_{i=1}^{3}\mathbf{q}_i\right)a^2(\tau) && (4.67)\\
&= \frac{E}{2}(2\pi)^3\int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}\varphi(\mathbf{q}_1,\tau)\varphi(\mathbf{q}_2,\tau)\varphi(\mathbf{q}_3,\tau)\\
&\times\frac{(-1)}{2}\left([\mathbf{q}_2+\mathbf{q}_3]^2-q_2^2-q_3^2\right)\delta^{(3)}\left(\sum_{i=1}^{3}\mathbf{q}_i\right)a^2(\tau) && (4.68)\\
&= \frac{E}{4}(2\pi)^3\int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}\varphi(\mathbf{q}_1,\tau)\varphi(\mathbf{q}_2,\tau)\varphi(\mathbf{q}_3,\tau)\\
&\times\left(q_2^2+q_3^2-q_1^2\right)\delta^{(3)}\left(\sum_{i=1}^{3}\mathbf{q}_i\right)a^2(\tau) && (4.69)\\
&\Longrightarrow\\
B_{\varphi(\nabla\varphi)^2} &= \frac{-2E}{4}\times\mathrm{Im}\left[\prod_{a=1}^{3}f^*_{k_a}(\tau)\right]\int_{-\infty}^{\tau}\frac{d\tau'}{(H\tau')^2}\\
&\times\sum_{\text{perms}}f_{k_1}(\tau')f_{k_2}(\tau')f_{k_3}(\tau')\left(k_2^2+k_3^2-k_1^2\right) && (4.70)\\
&= -\frac{E}{2}\left(\frac{H}{\sqrt{2c_s(k_1k_2k_3)}}\right)^{3\times2}\mathrm{Im}\Bigg[\int_{-\infty}^{\tau}\frac{d\tau'}{(H\tau')^2}e^{-ic_sk_T\tau'}\\
&\times\sum_{\text{perms}}(1+ic_sk_1\tau')(1+ic_sk_2\tau')(1+ic_sk_3\tau')\\
&\times\left(k_2^2+k_3^2-k_1^2\right)\left(1-ic_sk_T\tau+O(\tau^2)\right)\Bigg] && (4.71)\\
&= -\frac{E}{16}\frac{H^4}{c_s^3(k_1k_2k_3)^3}\mathrm{Im}\left[\left(1-ic_sk_T\tau+O(\tau^2)\right)\sum_{\text{perms}}G(k_1,k_2,k_3)\right] && (4.72)
\end{aligned}
$$

where

$$G(k_1, k_2, k_3) \equiv \int_{-\infty}^{\tau} d\tau' e^{-ic_s k_T \tau'} \left(k_2^2 + k_3^2 - k_1^2\right) \times \left((\tau')^{-2} + ic_s k_T (\tau')^{-1} - c_s^2 \sum_{a<b} k_a k_b - ic_s^3 k_1 k_2 k_3 \tau' \right) \tag{4.73}$$

$$= \left(k_2^2 + k_3^2 - k_1^2\right) \left(-\frac{e^{-ik_T c_s \tau}}{\tau} + c_s^2 \sum_{a<b} k_a k_b \frac{-i}{c_s k_T} + ic_s^3 k_1 k_2 k_3 \frac{(-i)^2}{(c_s k_T)^2} \right) \tag{4.74}$$

Hence,

$$\lim_{\tau \to 0} B_{\varphi(\nabla\varphi)^2} = -\frac{E}{16} \frac{H^4}{c_s^2 (k_1 k_2 k_3)^3} \sum_{\text{perms}} \left(k_2^2 + k_3^2 - k_1^2\right) \left(k_T - \sum_{a<b} k_a k_b / k_T - \frac{k_1 k_2 k_3}{k_T^2} \right) \tag{4.75}$$

$$= \frac{P_{X\phi}}{8} \frac{H^4}{c_s^2 (k_1 k_2 k_3)^3 k_T^2} \left(\sum k_a^2 \right) \left(k_T^3 - \sum_{a<b} k_a k_b k_T - k_1 k_2 k_3 \right) \tag{4.76}$$

So, in the far future, $\tau \to 0$

$$\boxed{B_{\varphi(\nabla\varphi)^2} = \frac{P_{X\phi}(P_X + 2P_{XX}\bar{X})^2}{8P_X^2} \frac{H^4}{(k_1 k_2 k_3)^3 k_T^2} \left(\sum k_a^2 \right) \left(k_T^3 - \sum_{a<b} k_a k_b k_T - k_1 k_2 k_3 \right)} \tag{4.77}$$

Bispectrum from $\dot{\varphi}\varphi^2$

$$H_{\dot{\varphi}\varphi^2} = D \int d^3 x a^4 \dot{\varphi}\varphi^2 \tag{4.78}$$

$$= (2\pi)^2 \delta^{(3)} \left(\sum \mathbf{q}_a\right) a^3 \int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3} \varphi'(\mathbf{q}_1,\tau)\varphi(\mathbf{q}_2,\tau)\varphi(\mathbf{q}_3,\tau) \tag{4.79}$$

$$\Longrightarrow$$

$$B_{\dot{\varphi}\varphi^2} = 2D\mathrm{Im} \sum_{\text{perms}} f^*_{k_1}(\tau) f^*_{k_2}(\tau) f^*_{k_3}(\tau) \int_{-\infty}^{\tau} \frac{d\tau'}{H^3(\tau')^3} f'_{k_1}(\tau') f_{k_2}(\tau') f_{k_2}(\tau') \tag{4.80}$$

$$= 2D \frac{H^3}{8c_s^3 (k_1k_2k_3)^3} \sum_{\text{perms}} \mathrm{Im} \left[\prod_{a=1}^{3}(1 - ik_a\tau)\right] \int_{-\infty}^{\tau} \frac{d\tau'}{(\tau')^2} e^{-ik_T(\tau'-\tau)}$$

$$\times \left[\prod_{b=2}^{3}(1 + ik_b\tau')\right] c_s^2 k_1^2 \tag{4.81}$$

As $\tau \to 0$, the featured integral

$$\int_{(1+ic_sk_2\tau')-\infty}^{\tau} e^{-ic_sk_T\tau'} \left(\frac{1}{(\tau')^2} + ic_s(k_2+k_3)\frac{1}{\tau'} - c_s^2 k_2 k_3\right)$$

is asymptotic to

$$-\frac{e^{-ic_sk_T\tau}}{\tau} - ik_1c_s \mathrm{Ei}(-ik_Tc_s\tau) - \frac{i}{k_T} c_s k_2 k_3$$

So, after cancelling terms and taking the imaginary part, we are left with

$$B_{\dot{\varphi}\varphi^2} \sim \frac{DH^3}{4c_s(k_1k_2k_3)^3} \sum_{\text{perms}} k_1^2 \left(c_sk_T - \frac{c_sk_2k_3}{k_T} - k_1c_s \mathrm{Re}(\mathrm{Ei}(-ik_Tc_s\tau))\right) \tag{4.82}$$

$$= \frac{DH^3}{2(k_1k_2k_3)^3} \left(k_T \sum k_a^2 - k_1k_2k_3 - \mathrm{Re}(\mathrm{Ei}(-ik_Tc_s\tau)) \sum k_a^3\right) \tag{4.83}$$

$$\sim -\frac{DH^3}{2(k_1k_2k_3)^3} \left(\sum k_a^3 (\gamma_E - 1 + \log|k_Tc_s\tau|) + k_1k_2k_3 - \sum_{a\neq b} k_a^2 k_b\right) \tag{4.84}$$

where to get the last line we have used the asymptotic expansion of the Exponential function. Using the expression (4.44),

$$\boxed{B_{\dot{\varphi}\varphi^2} \sim \frac{P_{X\phi}\dot{\bar{\phi}}H^3}{4(k_1k_2k_3)^3}\left(\sum k_a^3(\gamma_E - 1 + \log|k_T c_s \tau|) + k_1k_2k_3 - \sum_{a\neq b} k_a^2 k_b\right)} \tag{4.85}$$

Bispectrum from $\dot{\varphi}^2\varphi$

$$H_{\dot{\varphi}\varphi^2} = \lambda \int d^3x a^4 \dot{\varphi}^2 \varphi \tag{4.86}$$

$$\implies B_{\dot{\varphi}\varphi^2} = -2\lambda \mathrm{Im}\left(\sum_{\text{perms}} f^*_{k_1}(\tau) f^*_{k_2}(\tau) f^*_{k_3}(\tau) \int_{-\infty}^{\tau} d\tau' a^2 f'_{k_1}(\tau') f'_{k_2}(\tau') f'_{k_3}(\tau')\right) \tag{4.87}$$

$$\sim -2\lambda \frac{H^4 c_s^4}{8c_s^3(k_1k_2k_3)^3} \sum_{\text{perms}} k_1^4 k_2^4 \mathrm{Im} \int_{-\infty}^{0} d\tau' e^{-ic_s k_T \tau'}(1 + ic_s k_3 \tau') \tag{4.88}$$

$$= \frac{\lambda H^4 c_s}{4(k_1k_2k_3)^3} \sum_{\text{perms}} k_1^4 k_2^4 \mathrm{Im}\left(\frac{-i}{k_T c_s} + \frac{-ic_s k_3}{(c_s k_T)^2}\right) \tag{4.89}$$

$$= -\frac{\lambda H^4}{4(k_1k_2k_3)^3 k_T^2} \sum_{\text{perms}} k_1^4 k_2^4 (k_T + k_3) \tag{4.90}$$

$$= -\frac{\lambda H^4}{4(k_1k_2k_3)^3 k_T^2} \left(\sum_{a\neq b} \left[k_T(k_a k_b)^4 + k_1k_2k_3(k_ak_b)^3\right]\right) \tag{4.91}$$

Hence, using the expression (4.46), in the limit $\tau \to 0$, we have:

$$\boxed{B_{\dot{\varphi}\varphi^2} = \frac{(P_{XX\phi}\dot{\bar{\phi}} + P_{X\phi})H^4}{8(k_1k_2k_3)^3 k_T^2} \left(\sum_{a\neq b} \left[k_T(k_a k_b)^4 + k_1k_2k_3(k_ak_b)^3\right]\right)} \tag{4.92}$$

Bispectrum from φ^3 From (4.93), we know that a cubic potential interacion $V = \mu\varphi^3$ gives rise to a correlator

$$B_{\varphi^3} \sim \frac{\mu H^2}{2(k_1k_2k_3)^3}\left[\sum_a k_a^3(\gamma_E - 1 + \log|k_T c_s \tau|) + k_1k_2k_3 - \sum_{a\neq b} k_a^2 k_b\right] \tag{4.93}$$

after putting the factors of c_s where it corresponds.
For a $P(X,\phi)$, we have $\mu = P_{\phi\phi\phi}/6$ (clear from (4.36)), so the corresponding correlator is

$$\boxed{B_{\varphi^3} \sim \frac{P_{\phi\phi\phi} H^2}{12(k_1k_2k_3)^3}\left[\sum_a k_a^3(\gamma_E - 1 + \log|k_T c_s \tau|) + k_1k_2k_3 - \sum_{a\neq b} k_a^2 k_b\right]} \tag{4.94}$$

Summary: All of the 3-point correlators scale us $1/k^6$. None of them depends on the orientation of the momenta $\mathbf{k}_i$, only on their moduli k_i.
The correlators from the interactions $\dot{\varphi}\varphi^2$ and φ^3 blow up, as $\log |k_T c_s \tau|$. All the others do not diverge. Note that

$$\dot{\varphi}\varphi^2 \propto \frac{d}{dt}\varphi^3 \tag{4.95}$$

which explains the divergence of the the correlator from $\dot{\varphi}\varphi^2$.

5 Calculate the trispectrum of some scalar with various interactions

5.1 Trispectrum for $\dot{\varphi}^4$

This trispectrum was calculated previously for the case $c_s = 1$, in section 4.1.1. For an interacting Hamiltonian given by

$$H_{int} = -\lambda \int d^3x \sqrt{-g}\dot{\varphi}^4 \tag{5.1}$$

the corresponding trispectrum, in the limit $\tau \to 0$, given by (4.21):

$$B_4 = \frac{3H^8 4! \lambda}{k_T^5 k_1 k_2 k_3 k_4} \tag{5.2}$$

For a general c_s,

$$B_4 = \frac{2\lambda 4! H^8}{16(k_1k_2k_3k_4)^3 c_s^4} c_s^8 (k_1k_2k_3k_4)^2 \mathrm{Im} \int_{-\infty}^{\tau} d\tau' e^{-ik_T c_s \tau'} (\tau')^4 \tag{5.3}$$

Hence, in the limit $\tau \to 0$,

$$\boxed{B_4 = \frac{3H^8 4! \lambda}{k_T^5 c_s k_1 k_2 k_3 k_4}} \tag{5.4}$$

5.2 Trispectrum from $\dot{\varphi}^3\varphi$

Consider the following interacting Hamiltonian

$$H_{int} = -\lambda \int d^3x \sqrt{-g}\varphi\dot{\varphi}^3 \tag{5.5}$$

Then the resulting trispectrum is

$$B_4 = 2\lambda \text{Im}\left(f^*_{k_1}(\tau) f^*_{k_2}(\tau) f^*_{k_3}(\tau) f^*_{k_4}(\tau) \sum_{\text{perms}} \int_{-\infty}^{\tau} \frac{-d\tau'}{H\tau'} f'_{k_1}(\tau') f'_{k_2}(\tau') f'_{k_3}(\tau') f_{k_4}(\tau') \right) \tag{5.6}$$

$$\to \frac{-\lambda H^7 c_s^2}{8(k_1k_2k_3k_4)^3} \sum_{\text{perms}} k_1^2 k_2^2 k_3^2 \text{Im} \int_{-\infty}^{0} d\tau' e^{-ic_s k_T \tau'} (\tau')^2 (1 + ic_s k_4 \tau') \tag{5.7}$$

$$= \frac{-\lambda H^7 c_s^2}{8(k_1k_2k_3k_4)^3} \sum_{\text{perms}} k_1^2 k_2^2 k_3^2 \text{Im} \left(-\frac{(-i)^3 2}{c_s^3 k_T^3} - ic_s k_4 \frac{(-i)^4 6}{c_s^4 k_T^4} \right) \tag{5.8}$$

$$= \frac{-3\lambda H^7}{2c_s (k_1k_2k_3k_4)^3 k_T^4} \left(\sum_{a<b<c} (k_a k_b k_c)^2 k_T - 3k_1k_2k_3k_4 \sum_{a<b<c} k_a k_b k_c \right) \tag{5.9}$$

5.3 Trispectrum from $\dot{\varphi}^2\varphi^2$

Consider the following interacting Hamiltonian

$$H_{int} = -\lambda \int d^3x \sqrt{-g} \varphi^2 \dot{\varphi}^2 \tag{5.10}$$

Then the resulting trispectrum is

$$B_4 = 2\lambda \text{Im}\left(f^*_{k_1}(\tau) f^*_{k_2}(\tau) f^*_{k_3}(\tau) f^*_{k_4}(\tau) \sum_{\text{perms}} \int_{-\infty}^{\tau} \frac{d\tau'}{(H\tau')^2} f'_{k_1}(\tau') f'_{k_2}(\tau') f_{k_3}(\tau') f_{k_4}(\tau') \right) \tag{5.11}$$

$$\to \frac{\lambda H^6}{8(k_1k_2k_3k_4)^3} \sum_{\text{perms}} k_1^2 k_2^2 \text{Im} \int_{-\infty}^{0} d\tau' e^{-ic_s k_T \tau'} (1 + ic_s k_3 \tau')(1 + ic_s k_4 \tau') \tag{5.12}$$

$$= \frac{\lambda H^6}{8c_s (k_1k_2k_3k_4)^3} \sum_{\text{perms}} k_1^2 k_2^2 \text{Im} \left(\frac{i}{k_T} + i\frac{k_3 + k_4}{k_T^2} - i\frac{k_3k_4}{k_T^3} \right) \tag{5.13}$$

$$= \frac{\lambda H^6}{8c_s (k_T k_1k_2k_3k_4)^3} \sum_{\text{perms}} k_1^2 k_2^2 (k_T^2 + k_T(k_3 + k_4) - k_3k_4) \tag{5.14}$$

$$= \frac{\lambda H^6}{2c_s (k_T k_1k_2k_3k_4)^3} \left[2k_T^2 \sum_{a<b} k_a^2 k_b^2 - k_T \sum_{a<b} k_a^2 k_b^2 (k_a + k_b) - k_1k_2k_3k_4 \sum_{a<b} k_a k_b \right] \tag{5.15}$$

5.4 Trispectrum from $\dot{\varphi}\varphi^3$

Consider the following interacting Hamiltonian

$$H_{int} = -\lambda \int d^3x \sqrt{-g}\varphi^3\dot{\varphi} \tag{5.16}$$

Then the resulting trispectrum is

$$B_4 = 2\lambda \mathrm{Im}\left(f^*_{k_1}(\tau)f^*_{k_2}(\tau)f^*_{k_3}(\tau)f^*_{k_4}(\tau) \sum_{\text{perms}} \int_{-\infty}^{\tau} \frac{-d\tau'}{(H\tau')^3} f'_{k_1}(\tau')f_{k_2}(\tau')f_{k_3}(\tau')f_{k_4}(\tau') \right) \tag{5.17}$$

$$\sim \frac{-2\lambda H^5 c_s^2}{16 c_s^4 (k_1k_2k_3k_4)^3} \sum_{\text{perms}} k_1^2 \mathrm{Im}\Bigg[\left(1 - ic_sk_T\tau + O(\tau^2)\right) \times \int_{-\infty}^{\tau} \frac{d\tau'}{(\tau')^2} e^{-ic_sk_T\tau'} \left(1 + ic_s(k_T - k_1)\tau' - c_s^2(\tau')^2 \sum_{1<a<b} k_ak_b - ic_s^3(\tau')^3 k_2k_3k_4 \right) \Bigg] \tag{5.18}$$

$$\sim \frac{-\lambda H^5}{8 c_s^2 (k_1k_2k_3k_4)^3} \sum_{\text{perms}} k_1^2 \mathrm{Im}\Bigg[\left(1 - ic_sk_T\tau + O(\tau^2)\right) \times \left(-\frac{e^{-ik_Tc_s\tau}}{\tau} - ic_sk_1 \mathrm{Ei}(-ik_Tc_s\tau) - \frac{ic_s^2}{c_sk_T} \sum_{1<a<b} k_ak_b + i\frac{c_s^3k_2k_3k_4}{c_s^2k_T^2} \right) \Bigg] \tag{5.19}$$

$$\sim \frac{-\lambda H^5}{8 c_s (k_1k_2k_3k_4)^3} \sum_{\text{perms}} k_1^2 \left(k_T - k_1 \mathrm{Re}\left[\mathrm{Ei}(-ik_Tc_s\tau)\right] - \sum_{1<a<b} k_ak_b/k_T + k_2k_3k_4/k_T^2 \right) \tag{5.20}$$

$$\sim \frac{-\lambda H^5}{8 k_T^2 c_s (k_1k_2k_3k_4)^3} \sum_{\text{perms}} k_1^2 \left(k_T^3 - k_1k_T^2\gamma_E - k_1k_T^2 \log|k_Tc_s\tau| - \sum_{1<a<b} k_ak_bk_T + k_2k_3k_4 \right) \tag{5.21}$$

$$= \frac{3\lambda H^5}{2 k_T c_s (k_1k_2k_3k_4)^3} \Big(k_T^2 \sum k_a^2 - \sum k_a^3 k_T(\gamma_E + \log|k_Tc_s\tau|) - \sum_{a<b<c} k_ak_bk_c(k_a + k_b + k_c) + k_1k_2k_3k_4 \Big) \tag{5.22}$$

5.5 Trispectrum from φ^4

Consider the following interacting Hamiltonian

$$H_{int} = -\lambda \int d^3x \sqrt{-g}\varphi^4 \tag{5.23}$$

Then the resulting trispectrum is

$$B_4 = 2\lambda \mathrm{Im}\left(f^*_{k_1}(\tau) f^*_{k_2}(\tau) f^*_{k_3}(\tau) f^*_{k_4}(\tau) \sum_{\text{perms}} \int_{-\infty}^{\tau} \frac{d\tau'}{(H\tau')^4} f_{k_1}(\tau') f_{k_2}(\tau') f_{k_3}(\tau') f_{k_4}(\tau') \right) \tag{5.24}$$

$$= \frac{2\lambda \times 4! H^4}{16 c_s^4 (k_1 k_2 k_3 k_4)^3} \mathrm{Im}\left(\left[\prod_{a=1}^{4} (1 - i k_a c_s \tau)\right] \int_{-\infty}^{\tau} d\tau' e^{-i c_s k_T \tau'} \left[\prod_{a=1}^{4} (1/\tau' + i k_a c_s)\right] \right) \tag{5.25}$$

In the limit $\tau \to 0$ the integral is asymptotic to

$$\begin{aligned}
&-\frac{1}{3\tau^3} + i k_T c_s \frac{2}{3}\left(-\frac{1}{2\tau^2}\right) \\
&+ \left[i k_T c_s \frac{1}{3}\left(-i k_T c_s\right) - c_s^2 \sum_{a<b} k_a k_b \right] \left(-\frac{1}{\tau}\right) \\
&+ \mathrm{Ei}(-i k_T c_s \tau) \left(-i c_s^3 \sum_{a<b<c} k_a k_b k_c - i k_T c_s^3 \left[k_T^2 \frac{1}{3} - \sum_{a<b} k_a k_b \right] \right) \\
&+ c_s^4 k_1 k_2 k_3 k_4 \frac{i}{k_T c_s}
\end{aligned} \tag{5.26}$$

$$\begin{aligned}
=& -\frac{1}{3\tau^3} - i \frac{k_T c_s}{3\tau^2} - \left[k_T^2 \frac{1}{3} - \sum_{a<b} k_a k_b \right] \left(\frac{c_s^2}{\tau} \right) \\
&- i c_s^3 \mathrm{Ei}(-i k_T c_s \tau) \left(\sum_{a<b<c} k_a k_b k_c + k_T \left[k_T^2 \frac{1}{3} - \sum_{a<b} k_a k_b \right] \right) \\
&+ i c_s^3 k_1 k_2 k_3 k_4 / k_T
\end{aligned} \tag{5.27}$$

Hence,

$$B_4 \sim \frac{\lambda \times 3H^4}{c_s(k_1k_2k_3k_4)^3}\left(k_T\left[k_T^2\frac{1}{3}-\sum_{a<b}k_ak_b\right]+k_1k_2k_3k_4/k_T\right.$$
$$\left.-(\gamma_E+\log|k_Tc_s\tau|)\left(\sum_{a<b<c}k_ak_bk_c+k_T\left[k_T^2\frac{1}{3}-\sum_{a<b}k_ak_b\right]\right)\right) \quad (5.28)$$

5.6 Trispectrum from $\varphi^2\partial_i\varphi\partial^i\varphi$

Consider the following interacting Hamiltonian

$$H_{int} = -\lambda\int d^3x\sqrt{-g}\varphi^2\partial_i\varphi\partial^i\varphi = -\lambda\int d^3x\sqrt{-g}\varphi^2\partial_i\varphi\partial_i\varphi a^{-2} \quad (5.29)$$

Then the resulting trispectrum is

$$B_4 = 2\lambda\mathrm{Im}\left(f^*_{k_1}(\tau)f^*_{k_2}(\tau)f^*_{k_3}(\tau)f^*_{k_4}(\tau)\sum_{\text{perms}}(-\mathbf{k}_1\cdot\mathbf{k}_2)\int_{-\infty}^{\tau}\frac{d\tau'}{(H\tau')^2}f_{k_1}(\tau')f_{k_2}(\tau')f_{k_3}(\tau')f_{k_4}(\tau')\right) \quad (5.30)$$

We calculate first the integral, taking the limit $\tau \to 0$:

$$\int_{-\infty}^{\tau}\frac{d\tau'}{(H\tau')^2}f_{k_1}(\tau')f_{k_2}(\tau')f_{k_3}(\tau')f_{k_4}(\tau') \quad (5.31)$$

$$=\frac{H^2}{\sqrt{2^4c_s^4(k_1k_2k_3k_4)^3}}$$
$$\times\int_{-\infty}^{\tau}d\tau' e^{-ik_Tc_s\tau'}\left(\frac{1}{(\tau')^2}+\frac{ic_sk_T}{\tau'}-c_s^2\sum_{a<b}k_ak_b-ic_s^3\tau'\sum_{a<b<c}k_ak_bk_c+c_s^4(\tau')^2k_1k_2k_3k_4\right) \quad (5.32)$$

$$=\frac{H^2}{\sqrt{2^4c_s^4(k_1k_2k_3k_4)^3}}\left[\frac{c_s^2(-i)}{c_sk_T}\sum_{a<b}k_ak_b-i\frac{c_s^3}{c_s^2k_T^2}\sum_{a<b<c}k_ak_bk_c-k_1k_2k_3k_4\frac{c_s^4(-i)^32}{c_s^3k_T^3}\right] \quad (5.33)$$

$$=\frac{-iH^2c_s}{\sqrt{2^4c_s^4(k_1k_2k_3k_4)^3}(k_T)^3}\left[k_T^2\sum_{a<b}k_ak_b+k_T\sum_{a<b<c}k_ak_bk_c+2k_1k_2k_3k_4\right] \quad (5.34)$$

Hence,

$$B_4 = -\frac{2\lambda H^6}{2^4 c_s^3 (k_1k_2k_3k_4k_T)^3} \sum_{\text{perms}} (-\mathbf{k}_1 \cdot \mathbf{k}_2) \left[k_T^2 \sum_{a<b} k_a k_b + k_T \sum_{a<b<c} k_a k_b k_c + 2k_1k_2k_3k_4 \right] \tag{5.35}$$

$$= -\frac{\lambda H^6}{4 c_s^3 (k_1k_2k_3k_4k_T)^3} \sum_{a<b} (-2\mathbf{k}_a \cdot \mathbf{k}_b) \left[k_T^2 \sum_{a<b} k_a k_b + k_T \sum_{a<b<c} k_a k_b k_c + 2k_1k_2k_3k_4 \right] \tag{5.36}$$

$$= -\frac{\lambda H^6}{4 c_s^3 (k_1k_2k_3k_4k_T)^3} \left[\sum k_a^2 \right] \left[k_T^2 \sum_{a<b} k_a k_b + k_T \sum_{a<b<c} k_a k_b k_c + 2k_1k_2k_3k_4 \right] \tag{5.37}$$

where the last line follows from (squaring) momentum conservation, ie

$$\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4 = 0 \tag{5.38}$$

5.7 Trispectrum from $\varphi\dot{\varphi}\partial_i\varphi\partial^i\varphi$

Consider the following interacting Hamiltonian

$$H_{int} = -\lambda \int d^3x \sqrt{-g}\, \dot{\varphi}\varphi\partial_j\varphi\partial^j\varphi = -\lambda \int d^3x a^2 \dot{\varphi}\varphi\partial_j\varphi\partial_j\varphi \tag{5.39}$$

Then the resulting trispectrum is

$$B_4 = 2\lambda \mathrm{Im} \sum_{\text{perms}} f_{k_1}^*(\tau) f_{k_2}^*(\tau) f_{k_3}^*(\tau) f_{k_4}^*(\tau) \int_{-\infty}^{\tau} (\mathbf{k}_2 \cdot \mathbf{k}_3) \frac{d\tau'}{H\tau'} f_{k_1}'(\tau') f_{k_2}(\tau') f_{k_3}(\tau') f_{k_4}(\tau') \tag{5.40}$$

$$\sim \frac{\lambda H^7}{8 c_s^2 (k_1k_2k_3k_4)^3} \mathrm{Im} \sum_{\text{perms}} k_1^2 (\mathbf{k}_2 \cdot \mathbf{k}_3) \int_{-\infty}^{0} d\tau' e^{-ik_T c_s \tau'} \prod_{a=2}^{4} (1 + i c_s k_a \tau') \tag{5.41}$$

$$= \frac{\lambda H^7}{8 c_s^3 (k_1k_2k_3k_4)^3 k_T^4} \sum_{\text{perms}} k_1^2 (\mathbf{k}_2 \cdot \mathbf{k}_3) \left(2k_T^3 - k_T^2 k_1 + 2k_T \sum_{1<a<b} k_a k_b + 6k_2k_3k_4 \right) \tag{5.42}$$

5.8 Trispectrum from $\dot{\varphi}^2\partial_i\varphi\partial^i\varphi$

Consider the following interacting Hamiltonian

$$H_{int} = -\lambda \int d^3x\sqrt{-g}\left(\partial_j\varphi\partial^j\varphi\right)\dot{\varphi}^2 = -\lambda \int d^3x a^2\dot{\varphi}\dot{\varphi}\partial_j\varphi\partial_j\varphi \qquad (5.43)$$

Then the resulting trispectrum is

$$B_4 = 2\lambda \mathrm{Im} \sum_{\text{perms}} f^*_{k_1}(\tau)f^*_{k_2}(\tau)f^*_{k_3}(\tau)f^*_{k_4}(\tau)\int_{-\infty}^{\tau}(-\mathbf{k}_3\cdot\mathbf{k}_4)d\tau' f'_{k_1}(\tau')f'_{k_2}(\tau')f_{k_3}(\tau')f_{k_4}(\tau') \qquad (5.44)$$

$$\sim \frac{-\lambda H^8}{8(k_1k_2k_3k_4)^3}\mathrm{Im}\sum_{\text{perms}} k_1^2k_2^2(\mathbf{k}_3\cdot\mathbf{k}_4)\int_{-\infty}^{0} d\tau' e^{-ik_Tc_s\tau'}(\tau')^2(1+ic_s(k_3+k_4)\tau' - c_s^2k_3k_4(\tau')^2) \qquad (5.45)$$

$$= \frac{-\lambda H^8}{8(k_1k_2k_3k_4)^3}\mathrm{Im}\sum_{\text{perms}} k_1^2k_2^2(\mathbf{k}_3\cdot\mathbf{k}_4)\left(-\frac{2i}{c_s^3k_T^3} - i(k_3+k_4)\frac{6}{c_s^3k_T^4} - ik_3k_4\frac{24}{c_s^3k_T^5}\right) \qquad (5.46)$$

$$= \frac{\lambda H^8}{4k_T^5(c_sk_1k_2k_3k_4)^3}\sum_{\text{perms}} k_1^2k_2^2(\mathbf{k}_3\cdot\mathbf{k}_4)\left(2k_T^2 + 6(k_3+k_4)k_T + 24k_3k_4\right) \qquad (5.47)$$

5.9 Trispectrum from $\partial_j\varphi\partial^j\varphi\partial_i\varphi\partial^i\varphi$

Consider the following interacting Hamiltonian

$$H_{int} = -\lambda \int d^3x\sqrt{-g}\partial_j\varphi\partial^j\varphi\partial_i\varphi\partial^i\varphi = -\lambda \int d^3x\partial_j\varphi\partial_j\varphi\partial_i\varphi\partial_i\varphi \qquad (5.48)$$

Then the resulting trispectrum is

$$
\begin{aligned}
B_4 =& 2\lambda \mathrm{Im}\left(f^*_{k_1}(\tau) f^*_{k_2}(\tau) f^*_{k_3}(\tau) f^*_{k_4}(\tau) \int_{-\infty}^{\tau} d\tau' f_{k_1}(\tau') f_{k_2}(\tau') f_{k_3}(\tau') f_{k_4}(\tau') \right) \\
& \times \sum_{\text{perms}} (\mathbf{k}_1 \cdot \mathbf{k}_2)(\mathbf{k}_3 \cdot \mathbf{k}_4) \qquad (5.49) \\
\sim & \frac{2\lambda H^8}{16 c_s^4 (k_1 k_2 k_3 k_4)^3} \mathrm{Im}\left(\int_{-\infty}^{0} d\tau' e^{-i c_s k_T \tau'} \left[\prod_{a=1}^{4} (1 + i k_a c_s \tau') \right] \right) \\
& \times \sum_{\text{perms}} (\mathbf{k}_1 \cdot \mathbf{k}_2)(\mathbf{k}_3 \cdot \mathbf{k}_4) \qquad (5.50)
\end{aligned}
$$

The integral is

$$
\begin{aligned}
& \int_{-\infty}^{0} d\tau' e^{-i c_s k_T \tau'} \left[\prod_{a=1}^{4} (1 + i k_a c_s \tau') \right] = \\
& \quad - \frac{-i}{c_s k_T} - i k_T c_s \frac{(-i)^2}{c_s^2 k_T^2} - i^2 c_s^2 \sum_{a<b} k_a k_b \frac{2(-i)^3}{c_s^3 k_T^3} \\
& \quad - i^3 c_s^3 \sum_{a<b<c} k_a k_b k_c \frac{6(-i)^4}{c_s^4 k_T^4} - i^4 c_s^4 k_1 k_2 k_3 k_4 \frac{24(-i)^5}{c_s^5 k_T^5} \qquad (5.51) \\
& = \frac{i}{c_s k_T^5} \left(2k_T^4 + 2k_T^2 \sum_{a<b} k_a k_b + 6 k_T \sum_{a<b<c} k_a k_b k_c + 24 k_1 k_2 k_3 k_4 \right) \qquad (5.52)
\end{aligned}
$$

Hence, in the limit, the trispectrum is

$$
\begin{aligned}
B_4 =& \frac{\lambda H^8}{k_T^5 c_s^5 (k_1 k_2 k_3 k_4)^3} \left(k_T^4 + k_T^2 \sum_{a<b} k_a k_b + k_T \sum_{a<b<c} k_a k_b k_c + 12 k_1 k_2 k_3 k_4 \right) \\
& \times [(\mathbf{k}_1 \cdot \mathbf{k}_2)(\mathbf{k}_3 \cdot \mathbf{k}_4) + (\mathbf{k}_1 \cdot \mathbf{k}_3)(\mathbf{k}_2 \cdot \mathbf{k}_4) + (\mathbf{k}_1 \cdot \mathbf{k}_4)(\mathbf{k}_3 \cdot \mathbf{k}_2)] \qquad (5.53)
\end{aligned}
$$

5.10 Summary

All of the 4-point correlators scale us $1/k^9$. The correlators from the interactions $\dot{\varphi}\varphi^3$ and φ^4 blow up, as $\log|k_T c_s \tau|$. All the others do not diverge. Note that

$$\dot{\varphi}\varphi^3 \propto \frac{d}{dt}\varphi^4 \tag{5.54}$$

which explains the divergence of the the correlator from $\dot{\varphi}\varphi^3$.
The correlators from interactions that do not involve spatial derivatives of φ do not depend on the orientation of the momenta $\mathbf{k}_i$, only on their moduli k_i.
Neither does the correlator from $\varphi^2\partial_i\varphi\partial^i\varphi$. However, the correlators from $\varphi\dot{\varphi}\partial_i\varphi\partial^i\varphi$, $\dot{\varphi}^2\partial_i\varphi\partial^i\varphi$ and $\partial_i\varphi\partial^i\varphi\partial_j\varphi\partial^j\varphi$ do depend on the relative orientation of the momenta.

6 Gravity

So far we have discussed a quantum scalar field in a classical, fixed spacetime. We will now also account for the quantum behaviour of gravity.
This section is again based on [17].

6.1 Effective Field Theories

In a *effective field theory*, we have a separation of scales $E \ll E_0$ where E_0 is the characteristic energy scale of our theory and E is the energy at which we make experiments. We choose a *cutoff* Λ such that is close, but below, E_0. We can Taylor expand the *effective action* at low energies,

$$S_\Lambda(\phi_L) = \int d^4x \sum g_a \mathcal{O}_a \tag{6.1}$$

If the mass dimesion of $\mathcal{O}_a$ is Δ_a, it can be shown (see [17]) that the term in the action corresponding to $g_a \mathcal{O}_a$ is of order

$$\lambda_a \left(\frac{E}{\Lambda}\right)^{\Delta_a - 4}$$

for some dimensionless constant λ_a. Operators with $\lambda > 4$ are hence very small at low energies $E \ll \Lambda$ (and are called *irrelevant*).
Hence, we only need to consider a finite number of operators (depending on the required acuracy).

6.1.1 Gravity as an Effective Field Theory

To quantize gravity, we include small perturbations around some fixed background:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{6.2}$$

The relevant cutoff turns out to be M_{Pl}, the Planck mass. So we will be able to quantize gravity and make predictions at ebergies well-below the Planck scale.

6.2 Constraint from the ADM formalism

We saw in section 3.3 that GR has only two degrees of freedom. To make these constraints appear more explicitly, we use the ADM formalism, writing the most generic line element as

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{6.3}$$

Here h_{ij} is a 3-dimensional metric, and we have introduces the *lapse* $N(x)$ and the *shift* $N^i(x)$.
The full metric is then decomposed into time-time, time-space and space-space parts

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_i N^i & N_i \\ N_i & h_{ij} \end{pmatrix} \implies g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N_i/N^2 \\ N_i/N^2 & h^{ij} - N^i N^j/N^2 \end{pmatrix} \tag{6.4}$$

The determinant of the metric is given by $\sqrt{-g} = \sqrt{h}N$.
We define

$$n_\mu = (-N, 0, 0, 0) \tag{6.5}$$

which is normalised, and the *extrinsic curvature* as the change of its spatial components $K_{ij} = \nabla_j n_i$. It is not very hard to show that

$$\boxed{K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - {}^{(3)}\nabla_{(i}N_{j)}\right)} \tag{6.6}$$

where ${}^{(3)}\nabla_i$ is the 3-dimensional covariant derivative corresponding to h_{ij}.
The *Gauss-Codazzi equation* is

$$R = {}^{(3)}R + (K_{ij}K^{ij} - K^2) - 2\nabla_\alpha\left(n^\beta \nabla_\beta n^\alpha - n^\alpha \nabla_\beta n^\beta\right) \tag{6.7}$$

which relates the 4d and 3d Ricci scalars. The Hilbert-Einstein action (1.10) can be then re-written as

$$S_0 = \frac{M_{Pl}^2}{2}\int d^4x\sqrt{h}N\left[{}^{(3)}R + K_{ij}K^{ij} - K^2\right] \tag{6.8}$$

while the $P(X, \phi)$ action in the ADM formalism is

$$S_P = \int d^3x dt N\sqrt{h}P(X, \phi) \tag{6.9}$$

Adding S_0 and S_P gives the full action S. Varying S gives the *constraint equations* [21]:

$$\frac{\delta S}{\delta N} = 0 \implies \boxed{{}^{(3)}R - (K_{ij}K^{ij} - K^2) + \frac{2}{M_{Pl}^2}[P - 2P_X(X + h^{ij}\partial_i\phi\partial_j\phi)] = 0} \tag{6.10}$$

$$\frac{\delta S}{\delta N^i} = 0 \implies \boxed{\nabla_j[K_i^j - \delta_i^j K] + \frac{2P_X}{M_{Pl}^2 N}\partial_i\phi\left(N^j\partial_j\phi - \dot{\phi}\right) = 0} \tag{6.11}$$

where we have used $X = -\frac{1}{2}\partial_\mu\phi\partial_\nu\phi$ and thus X does depend on both N, N^i:

$$\frac{\delta X}{\delta N^j} = -\frac{1}{N^2}\partial_j\phi(\dot{\phi} - N^i\partial_i\phi)\delta N^j \tag{6.12}$$

6.3 Scalar-Vector-Tensor decomposition

The metric perturbation $h_{\mu\nu}$ is a symmetric 4x4 matrix with 10 independent entries. These can be separated into rotation scalars, rotation-vectors, and rotation tensors with the definitions:

$$h_{i0} = N_i \equiv a^2\partial_i\psi + N_i^V \tag{6.13}$$

$$h_{ij} \equiv a^2[\delta_{ij}A + \partial_{ij}B + \partial_{(i}C_{j)} + \gamma_{ij}] \tag{6.14}$$

Here, the rotation-vectors are also transverse

$$\partial_i N_i^V = 0 = \partial_i C_i \tag{6.15}$$

as well as the rotation-tensor, which is also traceless

$$\gamma_{ii} = \partial_i\gamma_{ij} = 0 \tag{6.16}$$

Crucially, rotation-scalars, transverse rotation-vectors and transverse traceless rotation-tensors decouple from each other at linear order. This implies that to solve the equations of motion for one, we can set the others to zero. After finding the separate solutions, one can add them up.

6.4 Gauge transformations

We will consider *gauge transformations*

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \tag{6.17}$$

We will transform full tensors covariantly as usual, keeping the background unchanged. After dropping the prime from the new coordinates, we will attribute all the transformation to the permutations.
Examples:

- For a scalar field $\phi(x) = \phi(\bar{t}) + \varphi(x)$, the transformation to linear order is $\Delta\varphi = -\epsilon^\mu\partial_\mu = -\epsilon^0\dot{\bar{\phi}}$

- For tensors, $\Delta h_{\mu\nu} = -2\nabla_{(\mu}\epsilon_{\nu)}$

To study how the SVT components (6.13)-(6.14) transform, we also SVT-decompose the gauge parameter:

$$\epsilon^\mu = (\epsilon^0, \partial^i \epsilon^S + \epsilon^i_V) \tag{6.18}$$

where $\partial_i \epsilon^i_V = 0$.
Assuming that $\epsilon^\mu(x)$ vanishes as $\|\mathbf{x}\| \to \infty$, one can invert Laplacians and obtain the following linear gauge transformations of the SVT components for the metric:

$$\Delta A = 2H\epsilon_0 \tag{6.19}$$

$$\Delta B = -\frac{2}{a^2}\epsilon^S \tag{6.20}$$

$$\Delta C_i = -\frac{1}{a^2}\epsilon^V_i \tag{6.21}$$

$$\Delta \gamma_{ij} = 0 \tag{6.22}$$

$$\Delta h_{00} = -2\delta N = 2\dot{\epsilon}^0 \tag{6.23}$$

$$\Delta \psi = \frac{1}{a^2}(-\epsilon_0 - \dot{\epsilon}^S + 2H\epsilon^S) \tag{6.24}$$

$$\Delta N^V_i = -\dot{\epsilon}^V_i + 2H\epsilon^V_i \tag{6.25}$$

6.4.1 Different gauges

We will see two commonly used choices of *small gauge transformations* when studying inflation.

Spatially-flat gauge Here the spatial part of the metric is free from any scalar perturbation. Namely,

$$A = 0 = B \implies g_{ij} = a^2(\delta_{ij} + \gamma_{ij}) \tag{6.26}$$

which has only tensor perturbations. (When tensors are neglected, this is just the metric of flat FLRW background.)

Comoving gauge It is also called the ζ-gauge, and it is given by

$$\varphi = 0 = B \implies \tag{6.27}$$

$$ds^2 = (-1 - 2\delta N)dt^2 + 2N_i dx^i dt + a^2 dx^i dx^j [\delta_{ij}(1 + A) + \gamma_{ij}] \tag{6.28}$$

6.5 The bispectrum from inflation

6.5.1 Flat gauge and the decoupling limit

The constraints equations can be solved to first order in perturbation theory. Working in flat gauge (6.38), to linear order in scalar perturbations, one has

$$^{(3)}R = 0 \tag{6.29}$$
$$N = 1 + \delta N \tag{6.30}$$
$$N_i = a^2 \partial_i \phi \tag{6.31}$$
$$E_{ij} = a^2 (H\delta_{ij} - \partial_i \partial_j \psi) \tag{6.32}$$
$$E = 3H - \partial_i \partial_i \psi \tag{6.33}$$

where $E_{ij} \equiv NK_{ij}$ and $E = E^i_i$.
Expanding the constraints in (6.10)-(6.11) to linear order, one can solve for δN and ψ. Using repeatedly the background equations of motion (1.61)-(1.63), the solution can be written as

$$\partial_i \partial_i \psi = -\frac{\epsilon}{c_s^2} \partial_t \left(\frac{H\varphi}{\dot{\phi}} \right) \tag{6.34}$$
$$\delta N = \epsilon H \frac{\varphi}{\dot{\phi}} \tag{6.35}$$

telling us how spacetime is deformed by the presence of the scalar field perturbations φ (to linear order). This interactions of φ, induced by gravity, are slow-roll supressed.
Hence, the leading interactions are those coming from the scalar action $P(X, \phi)$ on a fixed-spacetime background.

6.5.2 Curvature perturbations

We define ζ to be the gauge-invariant quantity that in the comoving 6.4.1 gauge appears in the (spatial part of the) metric as

$$g_{ij} = a^2 e^{2\zeta} \delta_{ij} \tag{6.36}$$

To first order:

- $A = 2\zeta$ (in the comoving gauge)

- Hence, in a general gauge,

$$\zeta = \frac{A}{2} - \frac{H}{\dot{\phi}}\varphi \tag{6.37}$$

- In particular, in the flat gauge,

$$\zeta = -\frac{H}{\dot{\phi}}\varphi \tag{6.38}$$

To compute the power spectrum of ζ, defined analogously to (3.10), we only need first order terms, and so

$$\boxed{P_\zeta(k) = \frac{H^2}{\dot{\phi}P_X}P_{\varphi_c}(k) = \frac{1}{4\epsilon c_s}\left(\frac{H}{M_{Pl}^2}\right)^2\frac{1}{k^3}} \tag{6.39}$$

where we have used the power spectrum of φ_{c_s} (4.34), the definition of ϵ (1.37) and the acceleration equation for a P theory, (1.62).
For the bispectrum, however, we need second order corrections.
It can be shown (see [20]) that, at second order,

$$\zeta = -H\frac{\varphi}{\dot{\phi}} + \text{terms that are slow-roll suppressed or vanish when } \tau \to 0 \tag{6.40}$$

We can then write

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle \approx -\left(\frac{H}{\dot{\phi}}\right)^3\langle\varphi(\mathbf{k}_1)\varphi(\mathbf{k}_2)\varphi(\mathbf{k}_3)\rangle + O(\epsilon, \eta, \dots) \tag{6.41}$$

for $\tau \to 0$.
Both the power spectrum and the bispectrum of ζ are gauge-invariant quantities .

7 Maldacena's paper calculations

In this section I include derivations of the results from Maldacena's paper [20]. I will refer to the equations from the latest version of the paper in the arXiv,[22].

7.1 Derivation of the constraint equations in the comoving gauge

Working in a gauge with $\delta\phi = 0$, the action in the ADM formalism is

$$S = \frac{1}{2}\sqrt{h}[NR^{(3)} - 2NV + N^{-1}(E_{ij}E^{ij} - E^2) + N^{-1}\dot{\phi}^2] \tag{7.1}$$

where all dependance on N is already explicit. From this, it is trivial to obtain that setting $\frac{\delta S}{\delta N} = 0$ implies

$$\boxed{R^{(3)} - 2V - N^{-2}(E_{ij}E^{ij} - E^2) - N^{-2}\dot{\phi}^2 = 0} \tag{7.2}$$

To obtain the constraint equation arising from setting $\frac{\delta S}{\delta N^j} = 0$, it suffices to require that

$$0 = \delta \int \sqrt{h}N^{-1}(E_{ij}E^{ij} - E^2) \implies \tag{7.3}$$

$$= 2\int \sqrt{h}N^{-1}(E_{ij}\delta E^{ij} - E\delta E) \tag{7.4}$$

$$= 2\int \sqrt{h}N^{-1}(E^i_j\nabla_i\delta N^j - E\nabla_j\delta N^j) \tag{7.5}$$

$$= -2\int \sqrt{h}\delta N^j\nabla_i[N^{-1}(E^i_j - \delta^i_jE)] \tag{7.6}$$

for all variations δN^j. Hence, we obtain the other constraint equation,

$$\boxed{\nabla_i[N^{-1}(E^i_j - \delta^i_jE)] = 0} \tag{7.7}$$

7.2 Solving the constraints to first order in the comoving gauge

In the comoving gauge,

$$\delta\phi = 0 \tag{7.8}$$
$$h_{ij} = e^{2\rho}[(1+2\eta)\delta_{ij} + \gamma_{ij}] \tag{7.9}$$
$$\partial_i\gamma_{ij} = 0 \tag{7.10}$$
$$\gamma_{ii} = 0 \tag{7.11}$$

We want to solve the constraint equations to first order, so we will write

$$h_{ij} = e^{2\rho+2\zeta}(\delta_{ij} + \gamma_{ij}) \tag{7.12}$$

which correct to first order.
We write $N = 1 + \delta N$ and $N_i = e^{2\rho+2\zeta}\partial_i\psi$.
To first order, we have

$$E_{ij} = e^{2\rho+2\zeta}[(\dot{\rho}+\dot{\zeta})\delta_{ij} - \gamma_i\gamma_j\psi + \dots] \tag{7.13}$$

where the dots denote terms proportional to γ_{ij} or $\dot{\gamma_{ij}}$. Hence,

$$E = 3(\dot{\rho}+\dot{\zeta}) - \partial_i\partial_i\psi \implies \tag{7.14}$$
$$N^{-1}(E_{ij} - h_{ij}E) = e^{2\rho+2\zeta}[(1-\delta N)(-2\dot{\rho}\delta_{ij}) - 2\delta_{ij}\dot{\zeta} - \partial_i\partial_j\psi + \delta_{ij}\partial_k\partial_k\psi + \dots] \tag{7.15}$$
$$\implies \nabla^i[N^{-1}(E_{ij} - h_{ij}E)] = \partial_j(2\delta N\delta_{ij}\dot{\rho} - 2\delta_{ij}\dot{\zeta})e^{2\rho+2\zeta} \tag{7.16}$$

Hence,

$$\boxed{\delta N = \frac{\dot{\zeta}}{\dot{\rho}}} \tag{7.17}$$

To solve for the second constraint equation, we note that

$$R^{(3)} = e^{-2\rho-2\zeta}[-4\partial^2\zeta] \tag{7.18}$$

to first order on ζ. Using previous expressions, we find that, to first order,

$$E_{ij}E^{ij} = 3(\dot{\rho}+\dot{\zeta})^2 - 2\dot{\rho}\partial^2\psi \tag{7.19}$$
$$E^2 = 9(\dot{\rho}+\dot{\zeta})^2 - 6\dot{\rho}\partial^2\psi \tag{7.20}$$
$$E_{ij}E^{ij} - E^2 = -6(\dot{\rho}+\dot{\zeta})^2 + 4\dot{\rho}\partial^2\psi \tag{7.21}$$
$$= -6(\dot{\rho}^2 + 2\dot{\zeta}\dot{\rho}) + 4\dot{\rho}\partial^2\psi + \dots \tag{7.22}$$

Hence, to first-order, the LHS of (7.2) is given by

$$-4\left(\partial^2\zeta\right)e^{-2\rho-2\zeta}-2V-\left(1-\frac{2\dot{\zeta}}{\dot{\rho}}\right)(\dot{\phi}^2-6\dot{\rho}^2))-[-12\dot{\zeta}\dot{\rho}+4\dot{\rho}\partial^2\psi] \tag{7.23}$$

$$=-4\left(\partial^2\zeta\right)e^{-2\rho}+\frac{2\dot{\zeta}}{\dot{\rho}}\dot{\phi}^2+4\dot{\rho}\partial^2\psi \tag{7.24}$$

which must be equal to zero. Hence, to first order, we write

$$\boxed{\psi=-e^{-2\rho}\frac{\zeta}{\dot{\rho}}+\chi} \tag{7.25}$$

where

$$\boxed{\partial^2\chi=\frac{\dot{\phi}^2}{2\dot{\rho}^2}\dot{\zeta}} \tag{7.26}$$

7.3 2nd order action

7.3.1 Derivation

In order to find the quadratic action for ζ, we substitute the first order expressions in the action and expand it to second order.
As Maldacena says, we need not compute N or N^i to second order. We will look in detail at the explanation for N (the one for N^i is analogous):
We write $N=N_0+N_1+N_2$ where N_1 is the first order term and N_2 includes the terms of order 2 or higher. Hence,

$$S=\int L(N=N_0+N_1+N_2) \tag{7.27}$$

$$=\int L(N=N_0+N_1)+N_2\frac{\partial L}{\partial N}(N=N_0+N_1) \tag{7.28}$$

is correct to second order. When solving the constraint equations, we have required that

$$\frac{\partial L}{\partial N}(N=N_0+N_1)=0 \tag{7.29}$$

is correct to first order, and thus

$$N_2\frac{\partial L}{\partial N}(N=N_0+N_1)=0 \tag{7.30}$$

is correct to second order. Hence, to second order,

$$S = \int L(N = N_0 + N_1) \tag{7.31}$$

as claimed.
There is a similar trick that we will also use. Recall that we have $\partial_i\psi$ is the first order term of the expansion in all orders of N^i. Not only can we ignore the high order terms, but we also can ignore the terms of order $O\left(\psi^2\right)$ in the Lagrangian: To second order,

$$S = \int L(N^i = \partial_i\psi) \tag{7.32}$$

$$= \int \left[L(N^i = 0) + \partial_i\psi \frac{\delta L}{\delta N_i}(N^i = 0) + \frac{1}{2}(\partial_i\psi)^2 \frac{\delta^2 L}{(\delta N_i)^2}(N^i = 0) \right] \tag{7.33}$$

From the constraint equations, we have

$$\frac{\delta L}{\delta N_i}(N^i = \partial_i\psi) = 0 \implies 0 = \frac{\delta^2 L}{(\delta N_i)^2}(N^i = \partial_i\psi) \tag{7.34}$$

to first order, and hence,

$$\frac{\delta^2 L}{(\delta N_i)^2}(N^i = 0) = 0 \tag{7.35}$$

to zero-th order. So, to second order,

$$S = \int \left[L(N^i = 0) + \partial_i\psi \frac{\delta L}{\delta N_i}(N^i = 0) \right] \tag{7.36}$$

We will compute the action in the comoving gauge, using the metric h_{ij} in the form of equation (3.1) of Maldacena's paper. We have

$$\sqrt{h} = e^{3\rho+3\zeta} \tag{7.37}$$

$$R^{(3)} = e^{-2\rho-2\zeta}\left[-4\partial^2\zeta - 2(\partial\zeta)^2\right] \tag{7.38}$$

Including now terms quadratic on ζ, one gets

$$E_{ij}E^{ij} - E^2 = -6(\dot\rho + \dot\zeta)^2 + 4(\dot\rho + \dot\zeta)\nabla_i N^i + O\left(N_i^2\right) \tag{7.39}$$

We will neglect the term $O\left(N_i^2\right) = O\left(\psi^2\right)$ in the basis of the above discussion. Substituting this into the action, we get

$$\begin{aligned} S =& \frac{1}{2}\int e^{\rho+\zeta}\left(1+\frac{\dot{\zeta}}{\dot{\rho}}\right)\left[-4\partial^2\zeta - 2\left(\partial\zeta\right)^2 - 2Ve^{2\rho+2\zeta}\right] \\ &+ e^{3\rho+3\zeta}\frac{1}{\left(1+\frac{\dot{\zeta}}{\dot{\rho}}\right)}[-6(\dot{\rho}+\dot{\zeta})^2 + 4(\dot{\rho}+\dot{\zeta})\left(\nabla_i N^i\right) + \dot{\phi}^2] \end{aligned} \tag{7.40}$$

We will show that the following term vanishes:

$$\int e^{3\rho+3\zeta}\frac{(\dot{\rho}+\dot{\zeta})}{\left(1+\frac{\dot{\zeta}}{\dot{\rho}}\right)}\left(\nabla_i N^i\right) = \int e^{3\rho+3\zeta}\dot{\rho}\left(\nabla_i N^i\right) = \int \dot{\rho}\sqrt{h}\nabla_i N^i = 0 \tag{7.41}$$

Hence, the action is

$$\begin{aligned} S =& \frac{1}{2}\int e^{\rho+\zeta}\left(1+\frac{\dot{\zeta}}{\dot{\rho}}\right)\left[-4\partial^2\zeta - 2\left(\partial\zeta\right)^2 - 2Ve^{2\rho+2\zeta}\right] \\ &+ e^{3\rho+3\zeta}\frac{1}{\left(1+\frac{\dot{\zeta}}{\dot{\rho}}\right)}\left[-6(\dot{\rho}+\dot{\zeta})^2 + \dot{\phi}^2\right] \end{aligned} \tag{7.42}$$

7.3.2 Simplification

To simplify the action (7.42), we will split into two parts. Firstly,

$$\frac{1}{2}\int e^{3\rho+3\zeta}\left[\left(1+\frac{\dot{\zeta}}{\dot{\rho}}\right)(-2V) + \frac{-6(\dot{\rho}+\dot{\zeta})^2+\dot{\phi}^2}{\left(1+\frac{\dot{\zeta}}{\dot{\rho}}\right)}\right] \tag{7.43}$$

$$= \frac{1}{2}\int e^{3\rho+3\zeta}\left[\left(1+\frac{\dot{\zeta}}{\dot{\rho}}\right)(\dot{\phi}^2 - 6\dot{\rho}^2 - 6\dot{\rho}^2) + \dot{\phi}^2\left(1-\frac{\dot{\zeta}}{\dot{\rho}}+\frac{\dot{\zeta}^2}{\dot{\rho}^2}\right)\right] \tag{7.44}$$

$$= \frac{1}{2}\int e^{3\rho+3\zeta}\left[2\dot{\phi}^2 + \frac{\dot{\phi}^2}{\dot{\rho}^2}\dot{\zeta}^2 - 12\dot{\rho}(\dot{\rho}+\dot{\zeta})\right] \tag{7.45}$$

$$= \frac{1}{2}\int e^{3\rho+3\zeta}\left[-4\ddot{\rho} + \frac{\dot{\phi}^2}{\dot{\rho}^2}\dot{\zeta}^2 + 4\ddot{\rho}\right] = \frac{1}{2}\int e^{3\rho+3\zeta}\frac{\dot{\phi}^2}{\dot{\rho}^2}\dot{\zeta}^2 \tag{7.46}$$

where to get to the last line we have integrated by parts.
Secondly,

$$\frac{1}{2}\int e^{\rho+\zeta}\left(1+\frac{\dot{\zeta}}{\dot{\rho}}\right)\left[-4\partial^2\zeta-2(\partial\zeta)^2\right] \tag{7.47}$$

$$=\frac{1}{2}\int e^{\rho}\left(-4\partial^2\zeta\frac{\dot{\zeta}}{\dot{\rho}}-\zeta 4\partial^2\zeta-2(\partial\zeta)^2\right) \tag{7.48}$$

$$=\frac{1}{2}\int e^{\rho}\left(\frac{4}{\dot{\rho}}\partial\dot{\zeta}\partial\zeta+4(\partial\zeta)^2-2(\partial\zeta)^2\right) \tag{7.49}$$

$$=\frac{1}{2}\int e^{\rho}\left(2(\partial\zeta)^2-2(\partial\dot{\zeta})^2\left(1-\ddot{\rho}/\dot{\rho}^2\right)\right) \tag{7.50}$$

$$=\frac{1}{2}\int e^{\rho}\left(-\frac{\dot{\phi}^2}{\dot{\rho}^2}\right) \tag{7.51}$$

where to obtain the second line we have neglected a total derivative in the integrand, and we are always working to second order.
Altogether, we get

$$\boxed{S=\frac{1}{2}\int\frac{\dot{\phi}^2}{\dot{\rho}^2}\left[e^{3\rho}\dot{\zeta}^2-e^{\rho}\left(\partial\zeta\right)^2\right]} \tag{7.52}$$

7.4 Solving the constraints to first order in the flat gauge

It is easy to prove[9] that the constraint equations (7.2) and (7.7) in a gauge where $\phi-\bar{\phi}(t)=\delta\phi=\varphi(x,t)$ are given by

$$0=R^{(3)}-2V(\bar{\phi}+\varphi)-N^{-2}\left(E_{ij}E^{ij}-E^2\right)-N^{-2}\left(\dot{\bar{\phi}}+\dot{\varphi}-N^i\partial_i\varphi\right)^2-h^{ij}\partial_i\varphi\partial_j\varphi \tag{7.53}$$

$$\nabla_i[N^{-1}(E^i_j-\delta^i_jE)]=N^{-1}\partial_j\varphi(\dot{\bar{\phi}}+\dot{\varphi}-N^k\partial_k\varphi) \tag{7.54}$$

[9]See [21], although note that $X_{there}=2X_{here}$

In flat gauge, we have

$$h_{ij} = e^{2\rho}(\delta_{ij} + \gamma_{ij}) \tag{7.55}$$
$$\partial_i \gamma_{ij} = 0 \tag{7.56}$$
$$\gamma_{ii} = 0 \tag{7.57}$$
$$R^{(3)} = 0 \tag{7.58}$$

We write

$$N = 1 + \delta N \tag{7.59}$$
$$N_i = e^{2\rho}\partial_i \chi \tag{7.60}$$
$$\tag{7.61}$$

Hence, we have

$$E_{ij} = e^{2\rho}\left(\dot{\rho}(\delta_{ij} + \gamma_{ij}) + \dot{\gamma}_{ij}/2 - \partial_i\partial_j\chi\right) \tag{7.62}$$
$$E = 3\dot{\rho} - \partial_i\partial_i\chi \tag{7.63}$$

To first order, the constraint equation (7.54),

$$\begin{aligned}(\partial_j\varphi)\dot{\bar{\phi}} =& \nabla_i[(1-\delta N)(\dot{\rho}(1-3)\delta_{ij})] \\ &+ \partial_i(\dot{\rho}\gamma_{ij} + \dot{\gamma}_{ij}/2 - \partial_i\partial_j\chi + \delta_{ij}\partial_k\partial_k\chi) \qquad (7.64) \\ =& 2\dot{\rho}\partial_j\delta N \qquad (7.65)\end{aligned}$$

and thus

$$\boxed{\delta N = \frac{\dot{\bar{\phi}}}{2\dot{\rho}}\varphi} \tag{7.66}$$

To first order,

$$E_{ij}E^{ij} - E^2 = -6\dot{\rho}^2 + 4\dot{\rho}\partial_k\partial_k\chi \tag{7.67}$$

Hence, the RHS of the 1st constraint eq, (7.53), to first order is given by

$$\begin{aligned}&-2V(\bar{\phi}) - 2\varphi V'(\bar{\phi}) + 6\dot{\rho}^2 - 12\dot{\rho}^2\delta N - 4\dot{\rho}\partial_k\partial_k\chi + 2\delta N\dot{\bar{\phi}}^2 - 2\dot{\varphi}\dot{\bar{\phi}} \qquad (7.68) \\ &= -12\dot{\rho}^2\delta N - 4\dot{\rho}\partial_k\partial_k\chi + 2\delta N\dot{\bar{\phi}}^2 - 2\dot{\varphi}\dot{\bar{\phi}} - 2\varphi V'(\bar{\phi}) \qquad (7.69)\end{aligned}$$

after using the background equation of motion. Setting this equal to zero, we get

$$\partial_k\partial_k\chi = \frac{1}{2\dot{\rho}}\left[\delta N\dot{\bar{\phi}}^2 - \dot{\varphi}\dot{\bar{\phi}} - 6\dot{\rho}^2\delta N - \varphi V'\right] \tag{7.70}$$

$$= \frac{\dot{\bar{\phi}}^2}{2\dot{\rho}^2}\left[\frac{\dot{\bar{\phi}}\varphi}{2} - \frac{\dot{\rho}\dot{\varphi}}{\dot{\bar{\phi}}} - \frac{3\dot{\rho}^2}{\dot{\bar{\phi}}}\varphi - \frac{V'\dot{\rho}}{\dot{\bar{\phi}}^2}\varphi\right] \tag{7.71}$$

$$\frac{d}{dt}\left(-\frac{\dot{\rho}}{\dot{\bar{\phi}}}\varphi\right) = -\frac{\ddot{\rho}\varphi}{\dot{\bar{\phi}}} - \frac{\dot{\rho}\dot{\varphi}}{\dot{\bar{\phi}}} + \frac{\dot{\rho}\ddot{\bar{\phi}}}{\dot{\bar{\phi}}^2}\varphi \tag{7.72}$$

$$= \frac{\dot{\bar{\phi}}\varphi}{2} - \frac{\dot{\rho}\dot{\varphi}}{\dot{\bar{\phi}}} - \frac{3\dot{\rho}^2}{\dot{\bar{\phi}}}\varphi - \frac{V'\dot{\rho}}{\dot{\bar{\phi}}^2}\varphi \tag{7.73}$$

where in the last line we have used the background equations of motion. Hence,

$$\boxed{\partial_k\partial_k\chi = \frac{\dot{\bar{\phi}}^2}{2\dot{\rho}^2}\frac{d}{dt}\left(-\frac{\dot{\rho}}{\dot{\bar{\phi}}}\varphi\right)} \tag{7.74}$$

7.5 Cubic terms in the Lagrangian

We work in a gauge where

$$\delta\phi = 0 \tag{7.75}$$

$$h_{ij} = e^{2\rho+2\zeta}\hat{h}_{ij} \tag{7.76}$$

$$\det\left\{\hat{h}\right\} = 1 \tag{7.77}$$

$$\hat{h}_{ij} = \left(\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{il}\gamma_{lj}\right) \tag{7.78}$$

and $\gamma_{ii} = 0 = \partial_i\gamma_{ij}$ to second order.
I will compute the cubic term in the Lagrangian corresponding to three scalars. The remaining terms in the cubic Lagrangian can be computed similarly.

7.5.1 Cubic terms for three scalars

Now we will expand the action up to cubic order in ζ.
Using the previous argument 7.3.1, we see that it is only necessary to know

N or N^i up to first order.
We now get extra terms in the action.
Firstly, equation (7.44) is replaced by

$$\frac{1}{2}\int e^{3\rho+3\zeta}\left[\left(1+\frac{\dot{\zeta}}{\dot{\rho}}\right)(\dot{\phi}^2-6\dot{\rho}^2-6\dot{\rho}^2)+\dot{\phi}^2\left(1-\frac{\dot{\zeta}}{\dot{\rho}}+\frac{\dot{\zeta}^2}{\dot{\rho}^2}-\frac{\dot{\zeta}^3}{\dot{\rho}^3}\right)\right] \tag{7.79}$$

$$=\frac{1}{2}\int e^{3\rho+3\zeta}\frac{\dot{\phi}^2}{\dot{\rho}^2}\dot{\zeta}^2\left(1-\frac{\dot{\zeta}}{\dot{\rho}}\right) \tag{7.80}$$

Secondly, now we can't neglect the terms of order $O(N^i)$. In particular, we need to know $\nabla_i N^j$. To compute this, we work out[10] the Levi-Civita connection:

$$\Gamma^a_{bc}=\delta_{ac}\partial_b\zeta+\delta_{ab}\partial_c\zeta-\delta_{bc}\partial_a\zeta \tag{7.81}$$

Hence,

$$\nabla_i N^j=\partial_i\partial_j\psi+\delta_{ij}\partial_k\zeta\partial_k\psi+2\partial_{[i}\zeta\partial_{j]}\psi\nabla_i N^i \quad =\partial^2\psi+3\partial_k\zeta\partial_k\psi \tag{7.82}$$

Now, $E_{ij}E^{ij}$ gets an extra term of

$$\nabla_i N^j\nabla_j N^i=\partial_i\partial_j\psi\partial_i\partial_j\psi+\partial^2\psi\partial_k\zeta\partial_k+\dots \tag{7.83}$$

where the dots are quartic order terms. Similarly, E^2 gets an extra term of

$$\left(\nabla_i N^i\right)^2=\left(\partial^2\psi\right)^2+6\partial_k\zeta\partial_k\psi\partial^2\psi \tag{7.84}$$

Overall, $E_{ij}E^{ij}-E^2$ gets an extra term of

$$\partial_i\partial_j\psi\partial_i\partial_j\psi-\left(\partial^2\psi\right)^2-4\partial_k\zeta\partial_k\psi\partial^2\psi \tag{7.85}$$

which, in the integrand of the action, is divided by $(1+\dot{\zeta}/\dot{\rho})$. Consequently, to third order, the action gets an extra term of

$$\int e^{3\rho+3\zeta}\left[\frac{1}{2}\left(\partial_i\partial_j\psi\partial_i\partial_j\psi-\left(\partial^2\psi\right)^2\right)\left(1-\frac{\dot{\zeta}}{\dot{\rho}}\right)-2\partial_k\zeta\partial_k\psi\partial^2\psi\right] \tag{7.86}$$

With both these changes in place, we do obtain equation (3.7) of Maldacena's paper for the third order in ζ action .

[10] Note that this expression is correct only to linear order in *zeta*. Here we do not need the terms linear in γ, but this would have to be computed to obtain the remaining terms in the cubic Lagrangian.

7.6 Computation of three point functions

For this calculations, we will use the same techniques that the ones we used to compute correlators of φ.
We note that the classical modes are related by

$$\zeta_k^{cl} = \frac{\dot{\rho}}{\dot{\phi}} f_k \tag{7.87}$$

$$(\gamma_k^s)^{cl} = \frac{\sqrt{2}}{M_{Pl}} f_k \tag{7.88}$$

where it is clear that k here refers to the momentum, and not to an spatial index (because γ is a 2d tensor).
A note for this section: factors of $\dot{\rho}$ and $\dot{\phi}$ in time-independent expressions (such us the final results for the correlators) should be taken to be $\dot{\rho}_*, \dot{\phi}_*$, as in Maldacena's paper.

Dimensional analysis $[\dot{\rho}] = M^1$, $[\dot{\phi}] = M^2$. All the correlators must have mass dimension -9. The momentum conserving delta function contributes with -3, while the scale invariance dependance $\sim k^{-6}$ gives the remaining -6. So the overall factor in the correlators should be dimensionless. We will use this to restore back the factors of M_{pl}.

7.6.1 Three scalars correlator

From the action in equation (3.13) of Maldacena's paper:

$$S_3 = \int \frac{\dot{\phi}^4}{\dot{\rho}^4} e^{5\rho} \dot{\rho} \dot{\zeta}_c^2 \partial^{-2} \dot{\zeta}_c d^3x dt = \int \frac{\dot{\phi}^4}{\dot{\rho}^4} e^{6\rho} \dot{\rho} \dot{\zeta}_c^2 \partial^{-2} \dot{\zeta}_c d^3x d\tau \tag{7.89}$$

Hence, the three-point correlator is

$$\langle\zeta_c\zeta_c\zeta_c\rangle' = -2\frac{\dot\rho^6}{\dot\phi^6}\text{Im}\left(f^*_{k_1}f^*_{k_2}f^*_{k_3}\int_{-\infty}^{\tau}\frac{\dot\phi^4}{\dot\rho^4}a^{6-3}\dot\rho f'_{k_1}f'_{k_2}f'_{k_3}\frac{d\tau'}{(-k_1^2)} + \text{perms}\right) \tag{7.90}$$

$$= -2\frac{\dot\rho^6}{\dot\phi^2}\frac{H^6}{8k_1^3k_2^3k_3^3}\text{Im}\left(\int_{-\infty}^{0}\frac{(\tau')^3}{\dot\rho^3}a^3e^{-ik_T\tau'}k_2^2k_3^2d\tau' + \text{perms}\right) \tag{7.91}$$

$$= \frac{4}{8k_1^3k_2^3k_3^3}\frac{\dot\rho^6}{\dot\phi^2}\frac{\sum_{i<j}k_i^2k_j^2}{k_T} \tag{7.92}$$

where I have used, as usual, $a = -1/(H\tau)$. Suppose we have, in Fourier space, a field redefinition

$$\zeta(\mathbf{k}) = \zeta_c(\mathbf{k}) + \lambda(\zeta_c * \zeta_c)(\mathbf{k}) \tag{7.93}$$

where * denotes the convolution. Here $\lambda = O(H_{int})$, so we want to work to first order in λ. In this case, using the obvious shorthand,

$$\langle\zeta^3\rangle - \langle\zeta_c^3\rangle = \lambda\left[\int_{\mathbf{q}}\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{q})\zeta(\mathbf{k}_3-\mathbf{q})\rangle + 2\text{ cyclic}\right] \tag{7.94}$$

$$= \lambda\sum_{cyclic}\left[\int_{\mathbf{q}}\left(\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{q})\rangle\langle\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3-\mathbf{q})\rangle + \langle\zeta(\mathbf{k}_2)\zeta(\mathbf{q})\rangle\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_3-\mathbf{q})\rangle\right)\right] \tag{7.95}$$

Further,

$$\int_{\mathbf{q}}\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{q})\rangle\langle\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3-\mathbf{q})\rangle = \int_{\mathbf{q}}(2\pi)^6\frac{H^4\dot\rho^4}{4k_1^3k_2^3\dot\phi^4}\delta^{(3)}(\mathbf{k}_1+\mathbf{q})\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_3-\mathbf{q}) \tag{7.96}$$

$$= (2\pi)^3\frac{H^4\dot\rho^4}{4k_1^3k_2^3\dot\phi^4}\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) \tag{7.97}$$

Hence,

$$\langle\zeta^3\rangle' - \langle\zeta_c^3\rangle' = 2\lambda\frac{H^4\dot\rho^4}{4k_1^3k_2^3\dot\phi^4} + \text{cyclic} \tag{7.98}$$

$$= \lambda\frac{\dot\rho^4\dot\rho^4}{8k_1^3k_2^3k_3^3\dot\phi^4}4\sum_i k_i^3 \tag{7.99}$$

Similarly, an extra term in the field redefinition (7.93) of the form $\mu\partial^{-2}(\zeta_c\partial^2\zeta_c)$ in real space, where $\mu = O(H_{int})$. In Fourier space, this becomes

$$\frac{\mu}{k^2}\int_{\mathbf{q}}\zeta_c(\mathbf{k}-\mathbf{q})\zeta_c(\mathbf{q})q^2 \tag{7.100}$$

A very similar calculation to the above shows that it leads to an extra term in $\langle\zeta^3\rangle'$ equal to:

$$\frac{\mu\dot{\rho}^8}{4\dot{\phi}^4k_1^3k_2^3k_3^3}\sum_{i\neq j}k_ik_j^2 = 2\mu\frac{\dot{\rho}^8}{\dot{\phi}^4}\frac{1}{\prod_i(2k_i^3)}\sum_{i\neq j}k_ik_j^2 \tag{7.101}$$

The field redefinition that applies to our correlator has

$$\lambda = \frac{1}{2}\frac{\ddot{\phi}}{\dot{\phi}\dot{\rho}} + \frac{1}{8}\frac{\dot{\phi}^2}{\dot{\rho}^2} \tag{7.102}$$

$$\mu = \frac{1}{4}\frac{\dot{\phi}^2}{\dot{\rho}^2} \tag{7.103}$$

Using this expressions and adding up the contributions (7.92), (7.100), (7.101) gives:

$$\boxed{\langle\zeta_1\zeta_2\zeta_3\rangle' = \frac{\dot{\rho}^8}{\dot{\phi}^4}\frac{1}{\prod_i(2k_i^3)}\mathcal{A}} \tag{7.104}$$

where, time dependent quantities should be evaluated at horizon crossing and

$$\mathcal{A} = 2\frac{\ddot{\phi}}{\dot{\phi}\dot{\rho}}\sum k_i^3 + \frac{\dot{\phi}^2}{\dot{\rho}^2}\left[\frac{1}{2}\sum k_i^3 + \frac{1}{2}\sum_{i\neq j}k_ik_j^2 + \frac{4}{k_T}\sum_{i<j}k_i^2k_j^2\right] \tag{7.105}$$

as in [20].

With mass factors

$$\boxed{\langle\zeta_1\zeta_2\zeta_3\rangle' = \frac{\dot{\rho}^8}{M_{pl}^2\dot{\phi}^4}\frac{1}{\prod_i(2k_i^3)}\mathcal{A}} \tag{7.106}$$

because A has mass dimension +2. *Comparing Maldacena's expression, he is missing a factor of M_{pl}^2.*

7.6.2 Two scalars and a graviton correlator

Here the relevant action is

$$S_3 = \frac{1}{2}\int \frac{\dot\phi^2}{\dot\rho^2} e^{\rho}\gamma_{ij}\partial_i\zeta\partial_j\zeta dt = \frac{1}{2}\int \frac{\dot\phi^2}{\dot\rho^2} e^{2\rho}\gamma_{ij}\partial_i\zeta\partial_j\zeta d\tau \tag{7.107}$$

Hence, using $q^i\epsilon_{ij}(\mathbf{q}) = 0$ and symmetry of ϵ_{ij}, we get

$$\langle\gamma^s(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle' = -2! \times 2\frac{\dot\rho^2}{\dot\phi^2}\epsilon^s_{ij}(-\mathbf{k}_1)\mathrm{Im}\left(f^*_{k_1}f^*_{k_2}f^*_{k_3}k^i_2k^j_3\int_{-\infty}^0 a^2 f_{k_1}f_{k_2}f_{k_3}d\tau'\right) \tag{7.108}$$

$$= -\frac{\dot\rho^6}{\dot\phi^2}\frac{4k^i_2k^j_3}{8k_1^3k_2^3k_3^3}\epsilon^s_{ij}(-\mathbf{k}_1)\mathrm{Im}\left(\int_{-\infty}^0 \frac{d\tau}{\tau^2}e^{-ik_T\tau}\prod_i(1+ik_i\tau)\right) \tag{7.109}$$

where we note that the polarization tensors must be taken outside of the expectation value before performing the trick (4.6).
This gives:

$$\boxed{\langle\gamma^s(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle' = \frac{\dot\rho^6}{\dot\phi^2}\frac{k^i_2k^j_3}{8k_1^3k_2^3k_3^3}\epsilon^s_{ij}(-\mathbf{k}_1)(4I)} \tag{7.110}$$

where, as in Maldacena's paper,

$$I = -k_T + \frac{\sum_{i<j}k_1k_j}{k_T} + \frac{k_1k_2k_3}{k_T^2} \tag{7.111}$$

With mass factors

$$\boxed{\langle\gamma^s(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle' = \frac{\dot\rho^6}{M_{Pl}^2\dot\phi^2}\frac{k^i_2k^j_3}{8k_1^3k_2^3k_3^3}\epsilon^s_{ij}(-\mathbf{k}_1)(4I)} \tag{7.112}$$

so Maldacena is missing a factor of M_{Pl}^2.

7.6.3 Two gravitons and a scalar correlator

Using the field redefinition 3.18 of Maldacena's paper, and the action

$$S = \frac{1}{4}\int \frac{\dot\phi^2}{\dot\rho^2}\dot\rho a^6\dot\gamma_{ij}\dot\gamma_{ij}\partial^{-2}\dot\zeta_c d\tau d^3x \tag{7.113}$$

we obtain, as usual, the expression for the correlator

$$\langle\zeta_c(\mathbf{k}_1)\gamma^{s_2}(\mathbf{k}_2)\gamma^{s_3}(\mathbf{k}_3)\rangle' = -\frac{2}{4}\frac{4\dot{\rho}^{(6+1-3)}2!}{\prod_i(2k_i^3)}\epsilon_{ij}^{s_2}(-\mathbf{k}_2)\epsilon_{ij}^{s_3}(-\mathbf{k}_3)\text{Im}\left(\int_{-\infty}^{0} k_2^2k_3^2 d\tau e^{-ik_T\tau}\right) \tag{7.114}$$

$$= \frac{4\dot{\rho}^4}{\prod_i(2k_i^3)}\epsilon_{ij}^{s_2}(-\mathbf{k}_2)\epsilon_{ij}^{s_3}(-\mathbf{k}_3)\frac{k_2^2k_3^2}{k_T} \tag{7.115}$$

Now, we need to add the remaining terms coming from the field redefinition of ζ, which are

$$-\frac{1}{32}\langle(\gamma_{ij}*\gamma_{ij})(\mathbf{k}_1)\gamma^{s_2}(\mathbf{k}_2)\gamma^{s_3}(\mathbf{k}_3)\rangle' = -\frac{2^2}{32}2\frac{\dot{\rho}^4}{4k_2^3k_3^3}\epsilon_{ij}^2\epsilon_{ij}^3 \tag{7.116}$$

$$\frac{1}{16}\langle\mathcal{F}[\partial^{-2}\gamma_{ij}\partial^2\gamma_{ij}](\mathbf{k}_1)\gamma^{s_2}(\mathbf{k}_2)\gamma^{s_3}(\mathbf{k}_3)\rangle' = \frac{2^2}{16}\frac{\dot{\rho}^4}{4k_1^3k_2^3k_3^3}k_1(k_2^2+k_3^2)\epsilon_{ij}^2\epsilon_{ij}^3 \tag{7.117}$$

The final result is obtained by adding (7.115), (7.116) and (7.117):

$$\boxed{\langle\zeta_1\gamma_2^{s_2}\gamma_3^{s_3}\rangle' = \frac{\dot{\rho}^4}{\prod_i(2k_i^3)}\epsilon_{ij}^{s_2}(-\mathbf{k}_2)\epsilon_{ij}^{s_3}(-\mathbf{k}_3)\left[4\frac{k_2^2k_3^2}{k_T}-\frac{1}{2}k_1^3+\frac{1}{2}k_1(k_2^2+k_3^2)\right]} \tag{7.118}$$

which agrees with [20] except for the coefficient in front of k_1^3: This disagreement was noted in [23].

With mass factors

$$\boxed{\langle\zeta_1\gamma_2^{s_2}\gamma_3^{s_3}\rangle' = \frac{\dot{\rho}^4}{M_{Pl}^4\prod_i(2k_i^3)}\epsilon_{ij}^{s_2}(-\mathbf{k}_2)\epsilon_{ij}^{s_3}(-\mathbf{k}_3)\left[4\frac{k_2^2k_3^2}{k_T}-\frac{1}{2}k_1^3+\frac{1}{2}k_1(k_2^2+k_3^2)\right]} \tag{7.119}$$

which does agree with the factors in Maldacena.

7.6.4 Three gravitons correlator

We want to compute the correlator

$$\langle\gamma_{\mathbf{k}_1}^{s_1}\gamma_{\mathbf{k}_2}^{s_2}\gamma_{\mathbf{k}_3}^{s_3}\rangle' \tag{7.120}$$

The relevant action is[11]

$$S_3 = -\int dt d^3x \frac{e^{2\rho}}{8} \left[2\left(\partial_k \gamma_{ij}\right)\gamma_{kl}\partial_j\gamma_{li} - \left(\partial_k\gamma_{ij}\right)\gamma_{kl}\partial_l\gamma_{ij}\right] \tag{7.121}$$

$$= -\int d\tau d^3x \frac{a^2}{8} \left[2\left(\partial_k \gamma_{ij}\right)\gamma_{kl}\partial_j\gamma_{li} - \left(\partial_k\gamma_{ij}\right)\gamma_{kl}\partial_l\gamma_{ij}\right] \tag{7.122}$$

$$= -\int d\tau d^3x \frac{a^2}{8} \left[\gamma_{ij}\gamma_{kl}\partial_k\partial_l\gamma_{ij} - 2\gamma_{li}\partial_k\gamma_{ij}\partial_j\gamma_{kl}\right] \tag{7.123}$$

The first term of the action can be rewritten as

$$(2\pi)^3\delta^{(3)}\left(\sum \mathbf{q}_i\right)\int \frac{a^2}{8}d\tau \int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3} \mathbf{q}_2^k\mathbf{q}_2^l\gamma_{ij}(\mathbf{q}_1)\gamma_{ij}(\mathbf{q}_2)\gamma_{kl}(\mathbf{q}_3) \tag{7.124}$$

so its contribution to our correlator (7.120) is

$$\frac{1}{4}2^3\epsilon_{ij}^{s_1}(-\mathbf{k}_1)\epsilon_{ij}^{s_2}(-\mathbf{k}_2)\epsilon_{kl}^{s_3}(-\mathbf{k}_3)\mathrm{Im}\left(f_{k_1}^*f_{k_2}^*f_{k_3}^*\mathbf{k}_2^k\mathbf{k}_2^l\int_{-\infty}^0 d\tau' a^2 f_{k_1}f_{k_2}f_{k_3} + \text{perms}\right) \tag{7.125}$$

$$=(-2I)\frac{\dot{\rho}^4}{\prod_i(2k_i^3)}\sum_{\text{perms}}\left(\epsilon_{ij}^{s_1}(-\mathbf{k}_1)\epsilon_{ij}^{s_2}(-\mathbf{k}_2)\epsilon_{kl}^{s_3}(-\mathbf{k}_3)\mathbf{k}_2^k\mathbf{k}_2^l\right) \tag{7.126}$$

where I is as above, given by (7.111). Similarly, the second term of the action contributes with

$$(4I)\frac{\dot{\rho}^4}{\prod_i(2k_i^3)}\sum_{\text{perms}}\left(\epsilon_{ij}^{s_1}(-\mathbf{k}_1)\epsilon_{kl}^{s_2}(-\mathbf{k}_2)\epsilon_{li}^{s_3}(-\mathbf{k}_3)\mathbf{k}_1^k\mathbf{k}_2^j\right) \tag{7.127}$$

Using the obvious shorthand, we have:

$$\langle\gamma_{\mathbf{k}_1}^{s_1}\gamma_{\mathbf{k}_2}^{s_2}\gamma_{\mathbf{k}_3}^{s_3}\rangle' = (-4I)\frac{\dot{\rho}^4}{\prod_i(2k_i^3)}\sum_{\text{perms}}\left[\frac{1}{2}\left(\epsilon_{ij}^1\epsilon_{ij}^2\epsilon_{kl}^3k_2^kk_2^l\right) - \left(\epsilon_{ij}^1\epsilon_{kl}^2\epsilon_{li}^3k_1^kk_2^j\right)\right] \tag{7.128}$$

where I have integrated by parts and used the transversality condition (which holds to second order).

[11] Maldacena does not write it explicitly, but see [23]

The first part of the sum is equal to

$$\sum_{cyclic} \frac{1}{2} \left(\epsilon^1_{ij}\epsilon^2_{ij}\epsilon^3_{kl}k^k_2 k^l_2 + \epsilon^3_{ij}\epsilon^2_{ij}\epsilon^1_{kl}k^k_2 k^l_2\right) \tag{7.129}$$

$$= \sum_{cyclic} \frac{1}{2} \left(\epsilon^1_{ii'}\epsilon^2_{jj'}\epsilon^3_{jj'}k^i_2 k^{i'}_2 + \epsilon^3_{ii'}\epsilon^2_{jj'}\epsilon^1_{jj'}(k^i_1 + k^i_3)(k^{i'}_1 + k^{i'}_3)\right) \tag{7.130}$$

$$= \sum_{cyclic} \frac{1}{2} \left(\epsilon^1_{ii'}\epsilon^2_{jj'}\epsilon^3_{jj'}k^i_2 k^{i'}_2 + \epsilon^3_{ii'}\epsilon^2_{jj'}\epsilon^1_{jj'}k^i_1 k^{i'}_1\right) \tag{7.131}$$

$$= \sum_{cyclic} \epsilon^1_{ii'}\epsilon^2_{jj'}\epsilon^3_{jj'}k^i_2 k^{i'}_2 \tag{7.132}$$

where the first equality follows from momentum conservation, while the second equality uses the transversality condition.
Similarly, one can show that

$$-\sum_{perms} \left(\epsilon^1_{ij}\epsilon^2_{kl}\epsilon^3_{li}k^k_1 k^j_2\right) = \sum_{cyclic} \left(\epsilon^1_{ij}\epsilon^2_{lj}\epsilon^3_{lk}k^i_2 k^k_1 + \epsilon^1_{ik}\epsilon^2_{ij}\epsilon^3_{lj}k^k_2 k^l_1\right) \tag{7.133}$$

Hence, adding both terms of the sum in (7.128) leads to

$$\boxed{\langle\gamma^{s_1}_{\mathbf{k}_1}\gamma^{s_2}_{\mathbf{k}_2}\gamma^{s_3}_{\mathbf{k}_3}\rangle' = (-4I)\frac{\dot{\rho}^4}{\prod_i(2k_i^3)}\epsilon^{s_1}_{ii'}(-\mathbf{k}_1)\epsilon^{s_2}_{jj'}(-\mathbf{k}_2)\epsilon^{s_3}_{ll'}(-\mathbf{k}_3)t_{ijl}t_{i'j'l'}} \tag{7.134}$$

where t_{ijl} is as in [20], and I have recovered the full notation for the polarization tensors.

With mass factors

$$\boxed{\langle\gamma^{s_1}_{\mathbf{k}_1}\gamma^{s_2}_{\mathbf{k}_2}\gamma^{s_3}_{\mathbf{k}_3}\rangle' = (-4I)\frac{\dot{\rho}^4}{M^4_{Pl}\prod_i(2k_i^3)}\epsilon^{s_1}_{ii'}(-\mathbf{k}_1)\epsilon^{s_2}_{jj'}(-\mathbf{k}_2)\epsilon^{s_3}_{ll'}(-\mathbf{k}_3)t_{ijl}t_{i'j'l'}} \tag{7.135}$$

which does agree with Maldacena.

8 Spinor helicity formalism

This section is a summary of the first three chapters of [24].
Throughout this section, we use the mostly minus sign convention for the metric: (+,-,-,-).
Conservation of momentum:

$$\sum_i p_i = 0 \tag{8.1}$$

On shell conditions:

$$p_i^2 = 0 \tag{8.2}$$

For ingoing momenta, the Mandelstam variables are:

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{8.3}$$
$$t = (p_1 + p_4)^2 = (p_2 + p_3)^2 \tag{8.4}$$
$$u = (p_1 + p_3)^2 = (p_2 + p_4)^2 \tag{8.5}$$

To compute scattering amplitudes, we construct a general ansatz for the S-matrix and sculpt out the correct answer from simple physical criteria:

- **Dimensional Analysis:** Scattering amplitudes should have mass dimension consistent with the dimensionality of the coupling constants in the theory.

- **Lorentz Invariance:** Scattering amplitudes should be Lorenzt invariant.Eg;
  - A four-particle amplitude of scalars is a function of s, t, u.
  - When there are particles with spin, the amplitude should also be covariant under the little group.

- **Locality** Scattering amplitudes should have kinematic singularities which are consistent with factorization and unitarity. These singularities encode the underlying locality of thee theory. Eg,
  - A four-particle amplitude can have poles like $1/s$ but not $1/s^2$.

8.1 Spinor Helicity

The spinor helicty formalism maps the component of a four-vector into those of two-by-two matrix via

$$p_{\alpha\dot{\alpha}} = p_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} \tag{8.6}$$

where $\sigma^\mu = (1, \sigma)$ is a four-vector of Pauli matrices and the undotted and dotted indices transform under the usual spinor representations of the Lorentz group.

The only Lorentz invariant quantity which can be constructed from $p_{\alpha\dot{\alpha}}$ is its determinant,

$$\det\{p\} = p^\mu p_\mu = 0 \tag{8.7}$$

Hence, since it is non-zero but has vanishing determinant, it is a two-by-two matrix of rank at most one. Without loss of generality, we write it as the outer product of two two-component objects (called spinors):

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \tag{8.8}$$

λ_α and $\tilde{\lambda}_{\dot{\alpha}}$ are called, respectively, ''holomorphic" and ''anti-holomorphic" spinors, because of their transformation properties under the Lorentz group. For real momenta, $p_{\alpha\dot{\alpha}}$ is Hermitian, implying the reality condition:

$$\tilde{\lambda}_\alpha = \pm \lambda^*_{\dot{\alpha}} \tag{8.9}$$

Given two particles i and j, we define:

$$\langle ij \rangle = \lambda_{i\alpha} \lambda_{j\beta} \epsilon^{\alpha\beta} \tag{8.10}$$

$$[ij] = \tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_{j\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}} \tag{8.11}$$

where here the subindices i, j in the spinors are just labels (not referring to coordinates) and:

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{8.12}$$

which lower and raise spinor indices. This means that $\langle ij \rangle = -\langle ji \rangle$ and $[ij] = -[ji]$. Moreover, spinors satisfy the Schouten identity:

$$\langle ij \rangle \lambda_k + \langle ki \rangle \lambda_j + \langle jk \rangle \lambda_i = 0 \tag{8.13}$$

Any function of four-dimensional kinematic data can be written exclusively in terms of these objects (known as the angle and square brackets). For example,

$$s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j = \langle ij \rangle [ij] \tag{8.14}$$

which can be proven by expanding out the product of the final expression.
Finally, note that both the angle and the square brackets have mass dimension 1 (because they have the same mass dimension as momentum).
In my notation, $[\tilde{i}\tilde{j}] \equiv [\bar{i}\bar{j}] \equiv \langle ij \rangle$ and $\langle \bar{i}\bar{j} \rangle \equiv [ij]$.

Further notation (from Schwartz's book) We can write

$$\lambda^\alpha = p\rangle \tag{8.15}$$
$$\lambda_\alpha = \langle p \tag{8.16}$$
$$\tilde{\lambda}_{\dot{\alpha}} = p] \tag{8.17}$$
$$\tilde{\lambda}^{\dot{\alpha}} = [p \tag{8.18}$$

8.2 Little Group

Spinor helicity variables linearly realize the symmetries of the system. These are Lorentz invariance and the little group, which we now discuss.
The *little group* is the subset of Lorentz transformations that leave the momentum p_μ of a particle invariant.
A particle of momentum p_μ is described by an irreducible representation of the little group.
Under the little group, the spinor helicity variables transform so at to leave p_i unchanged, so

$$\lambda_i \to t_i \lambda_i \tag{8.19}$$
$$\tilde{\lambda}_i \to t_i^{-1} \tilde{\lambda}_i \tag{8.20}$$

For real momenta, the condition (8.9) implies that t_i is just a pure phase.
The little group covariance enters through an additional set of kinematic objects we have thus far ignored: the polarizations vectors

$$e^+_{\alpha\dot{\alpha}} = \frac{\eta_\alpha \tilde{\lambda}_{\dot{\alpha}}}{\langle \eta \lambda \rangle} \tag{8.21}$$
$$e^-_{\alpha\dot{\alpha}} = \frac{\tilde{\eta}_\alpha \lambda_{\dot{\alpha}}}{[\eta \lambda]} \tag{8.22}$$

where the $\pm$ superscripts label helicity.
The reference spinors η and $\tilde{\eta}$ are linearly independent of λ and $\tilde{\lambda}$ but are otherwise arbitrary.
By construction, these polarizations are transverse to the momenta, so

$$p^{\alpha\dot{\alpha}} e^{+}_{\alpha\dot{\alpha}} \propto [\tilde{\lambda}\tilde{\lambda}] = 0 \tag{8.23}$$

$$p^{\alpha\dot{\alpha}} e^{-}_{\alpha\dot{\alpha}} \propto \langle\lambda\lambda\rangle = 0 \tag{8.24}$$

Under the little group, the polarization vectors transform as

$$e^{+}_{\alpha\dot{\alpha}} \to t^{-2} e^{+}_{\alpha\dot{\alpha}} \tag{8.25}$$

$$e^{-}_{\alpha\dot{\alpha}} \to t^{2} e^{-}_{\alpha\dot{\alpha}} \tag{8.26}$$

A scattering amplitude will then be multilinear in the corresponding polarizations, so

$$A(1^{h_1} \dots n^{h_n}) = e^{h_1}_{\mu_1} \dots e^{h_n}_{\mu_n} A^{\mu_1 \dots \mu_n} \tag{8.27}$$

Hence, the scattering amplitude is little group covariant with weight

$$A(1^{h_1} \dots n^{h_n}) \to \prod_i t_i^{-2h_i} A(1^{h_1} \dots n^{h_n}) \tag{8.28}$$

This strongly constrains the form that a scattering amplitude can have, to all orders in perturbation theory. Explicitly, **the number of factors of $i\rangle$ and $\langle i$ minus the number of factors of $i]$ and $[i$ in the amplitude must be equal to 2 for a negative helicity gluon and -2 for a positive helicity gluon**. It can be shown that, to all orders,

$$A(1^{+}2^{+} \dots n^{+}) = 0 = A(1^{-}2^{-} \dots n^{-}) \tag{8.29}$$

for all n, and

$$A(1^{-}2^{+} \dots n^{+}) = 0 = A(1^{+}2^{-} \dots n^{-}) \tag{8.30}$$

for all $n > 3$.
Thus the non-vanishing amplitudes will have at least two negative and two positive helicities. Those with exactly two negative or exactly two positive helicities are called *maximum helicity violating* (MHV) amplitudes.

8.3 Bootstrapping Amplitudes

This section aims to enumerate all possible Lorentz invariant interactions among massless particles in four dimensions.

8.3.1 Three-particle Amplitudes

From momentum conservation, $p_1 + p_2 + p_3 = 0$, we have:

$$(p_1 + p_2)^2 = \langle 12\rangle[12] = p_3^2 = 0 \tag{8.31}$$
$$(p_2 + p_3)^2 = \langle 23\rangle[23] = p_1^2 = 0 \tag{8.32}$$
$$(p_3 + p_1)^2 = \langle 31\rangle[31] = p_2^2 = 0 \tag{8.33}$$

This gives two possible kinematic configurations:

- holomorphic:

$$[12] = [23] = [31] = 0 \implies \tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3 \tag{8.34}$$

- or anti-holomorphic:

$$\langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0 \implies \lambda_1 \propto \lambda_2 \propto \lambda_3 \tag{8.35}$$

In the holomorphic configuration, WLOG

$$A(1^{h_1}2^{h_2}3^{h_3}) = \langle 12\rangle^{n_3}\langle 23\rangle^{n_1}\langle 31\rangle^{n_2} \tag{8.36}$$

Imposing little group covariance (8.28), we get

$$-2h_1 = n_2 + n_3 \tag{8.37}$$
$$-2h_2 = n_3 + n_1 \tag{8.38}$$
$$-2h_3 = n_1 + n_2 \tag{8.39}$$

and hence

$$n_1 = h_1 - h_2 - h_3 \tag{8.40}$$
$$n_2 = h_2 - h_3 - h_1 \tag{8.41}$$
$$n_3 = h_3 - h_1 - h_2 \tag{8.42}$$

Since the angle and the square brackets both have mass dimension 1, the amplitude has mass dimension

$$[A(1^{h_1}2^{h_2}3^{h_3})] = n_1 + n_2 + n_3 = -(h_1 + h_2 + h_3) = -h \tag{8.43}$$

The assumption of locality implies that the three-particle amplitude has non-negative mass dimension hence $h \leq 0$ for the holomorphic configuration.

Using the analogous results for the anti-holomorphic configuration, we obtain a general formula for the three-particle amplitude of massless particles in four dimensions:

$$A(1^{h_1}2^{h_2}3^{h_3}) = \begin{cases} \langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1} & h \leq 0 \\ [12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2} & h \geq 0 \end{cases} \tag{8.44}$$

Note that this formula is up to a constant of proportionality, this expression just factors out the full Little group covariance.

Scalars:

- For identical scalars, all helicities h_i vanish, so the three-particle is a constant,
$$A(123) = \omega \tag{8.45}$$
- For multiple scalars,
$$A(1_a 2_b 3_c) = \omega_{abc} \tag{8.46}$$
where ω_{abc} is symmetric, since the states are bosons.

Vectors:

- For identical vectors, all the helicities are $h_i = \pm 1$, so the exponents n_i are odd-integers. Hence, the three-particle amplitude is odd under the exchange of any two external states, which violates bosonic statistics unless the amplitude is exactle zero. So the three-particle amplitude of photons is zero.
- For multiple species, we have
$$A(1_a^- 2_b^- 3_c^+) = f_{abc}\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 32\rangle} \tag{8.47}$$
$$A(1_a^+ 2_b^+ 3_c^-) = f_{abc}\frac{[12]^3}{[13][32]} \tag{8.48}$$
$$A(1_a^- 2_b^- 3_c^-) = f_{abc}\langle 12\rangle\langle 23\rangle\langle 31\rangle \tag{8.49}$$
$$A(1_a^+ 2_b^+ 3_c^+) = f_{abc}[12][23][31] \tag{8.50}$$
where f_{abc} is fully antisymmetric in its indices (so that the amplitude is even under the exchange of bosons).

Tensors For identical tensors, the helicities are $h_i = \pm 2$. Hence, the exponents n_i are even. Thus:

$$A(1^{--}2^{--}3^{++}) = \frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 32\rangle^2} \tag{8.51}$$

$$A(1^{++}2^{++}3^{--}) = \frac{[12]^6}{[13]^2[32]^2} \tag{8.52}$$

$$A(1^{--}2^{--}3^{--}) = \langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2 \tag{8.53}$$

$$A(1^{++}2^{++}3^{++}) = [12]^2[23]^2[31]^2 \tag{8.54}$$

8.3.2 Four-Particle Amplitudes

Extending our analysis to four-particle amplitudes will require locality, which is encoded in the singularity structure of an amplitude.
In the four-particle amplitude, simple poles like $1/s$ can arise, but not poles like $1/s^2$.
For the tree-level four-particle amplitude, this implies that

$$\lim_{s\to 0} sA_4 = A_3A_3 \tag{8.55}$$

and hence

$$[A_4] = 2[A_3] - 2 \tag{8.56}$$

relates the mass dimensions of the four-particle and three-particle amplitudes.

Scalars The three-particle amplitude has positive mass dimension

$$[A_3] \geq 0$$

So (8.56) implies that

$$[A_4] \geq -2 \tag{8.57}$$

Moreover, the tree amplitude should be a permutation invariant function of s, t and u with only simple poles. Enumerating all possible such functions (that would also agree with (8.55), we obtain

$$\phi^3 : A_4 = 1/s + 1/t + 1/u \tag{8.58}$$

$$\phi^4 : A_4 = 1 \tag{8.59}$$

$$(\partial\phi)^2\phi^2 : A_4 = s + t + u = 0 \tag{8.60}$$

$$(\partial\phi)^4 : A_4 = s^2 + t^2 + u^2 \tag{8.61}$$

$$(\partial\partial\phi)^2(\partial\phi)^2 : A_4 = s^3 + t^3 + u^3 \tag{8.62}$$

Vectors For vectors, $[A_3] = 1$, so (8.56) implies

$$[A_4] = 2[A_2] - 2 = 0 \tag{8.63}$$

ie, the amplitude is dimensionless.
The amplitude transforms as

$$A(1_a^- 2_b^- 3_c^+ 4_d^+) \to t_1^2 t_2^2 t_3^{-2} t_4^{-2} A(1_a^- 2_b^- 3_c^+ 4_d^+) \tag{8.64}$$

under the little group.
Factoring out the full little group of the amplitude, we consider the general ansatz

$$A(1_a^- 2_b^- 3_c^+ 4_d^+) = \langle 12\rangle^2 [34]^2 F(s,t,u) \tag{8.65}$$

where

$$F(s,t,u) = \frac{c_{st}}{st} + \frac{c_{tu}}{tu} + \frac{c_{us}}{us} \tag{8.66}$$

which is the most general little group invariant function with only simple poles and mass dimension -4 (the coefficients c's are dimensionless constants).

Demanding factorization on the s-channel (8.55), we get:

$$\lim_{s\to 0} sA(1_a^- 2_b^- 3_c^+ 4_d^+) = \langle 12\rangle^2[34]^2\frac{1}{t}(c_{st} - c_{us}) \tag{8.67}$$

$$= \sum_{h=\pm}\sum_e A(1_a^- 2_b^- P_e^h)A(3_c^+ 4_d^+ P_e^{-h}) \tag{8.68}$$

$$= \sum_e A(1_a^- 2_b^- P_e^+)A(3_c^+ 4_d^+ P_e^-) \tag{8.69}$$

$$= \sum_e f_{abe}f_{cde}\frac{\langle 12\rangle^3}{\langle P1\rangle\langle 2P\rangle}\frac{[34]^3}{[P3][4P]} \tag{8.70}$$

$$= \sum_e f_{abe}f_{cde}\frac{\langle 12\rangle^3}{\langle 1P\rangle\langle P2\rangle}\frac{[34]^3}{[3P][P4]} \tag{8.71}$$

$$= \sum_e f_{abe}f_{cde}\frac{\langle 12\rangle^3[34]^3}{-\langle 12\rangle[24]\langle 42\rangle[34]} \tag{8.72}$$

$$= \sum_e f_{abe}f_{cde}\frac{\langle 12\rangle^2[34]^2}{-[24]\langle 42\rangle} \tag{8.73}$$

$$= \sum_e f_{abe}f_{cde}\frac{\langle 12\rangle^2[34]^2}{-t} \tag{8.74}$$

$$= \sum_e f_{abe}f_{cde}\langle 12\rangle^2[34]^2\frac{1}{t} \tag{8.75}$$

$$\implies c_{st} - c_{us} = \sum_e f_{abe}f_{cde} \tag{8.76}$$

where we have defined $P = -(p_1 + p_2) = (p_3 + p_4)$ and used $t = -u$ (in the limit $s \to 0$), as well as (8.29) to go from the second line to the third line. Repeating this for t and u gives:

$$c_{st} - c_{us} = \sum_e f_{abe}f_{cde} \tag{8.77}$$

$$c_{tu} - c_{st} = \sum_e f_{bce}f_{ade} \tag{8.78}$$

$$c_{us} - c_{tu} = \sum_e f_{cae}f_{bde} \tag{8.79}$$

and hence:

$$\boxed{\sum_e (f_{abe}f_{cde} + f_{bce}f_{ade} + f_{cae}f_{bde}) = 0} \tag{8.80}$$

which is the Jacobi identity.

Tensors The mass dimension of the three-particle graviton amplitude is $[A_3] = 2$, so (8.56) implies that

$$[A_4] = 2[A_3] - 2 = 2 \tag{8.81}$$

The amplitude transforms as

$$A(1^{--}2^{--}3^{++}4^{++}) \to t_1^4 t_2^4 t_3^{-4} t_4^{-4} A(1^{--}2^{--}3^{++}4^{++}) \tag{8.82}$$

so factoring out the full little group weight of the amplitude, we construct the general ansatz

$$A(1^{--}2^{--}3^{++}4^{++}) = \langle 12\rangle^4 [34]^4 F(s,t,u) \tag{8.83}$$

Dimensional analysis tell us that F should have mass dimension -6, and locality implies that it should only have simple poles. Hence,

$$F(s,t,u) = \frac{c}{stu} \tag{8.84}$$

where c is a dimensionless constant.
This amplitude, $-{+}{+}$, is MHV and is the only non-zero amplitude

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