An Introduction To Mathematical Biology

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Women In Maths 2021

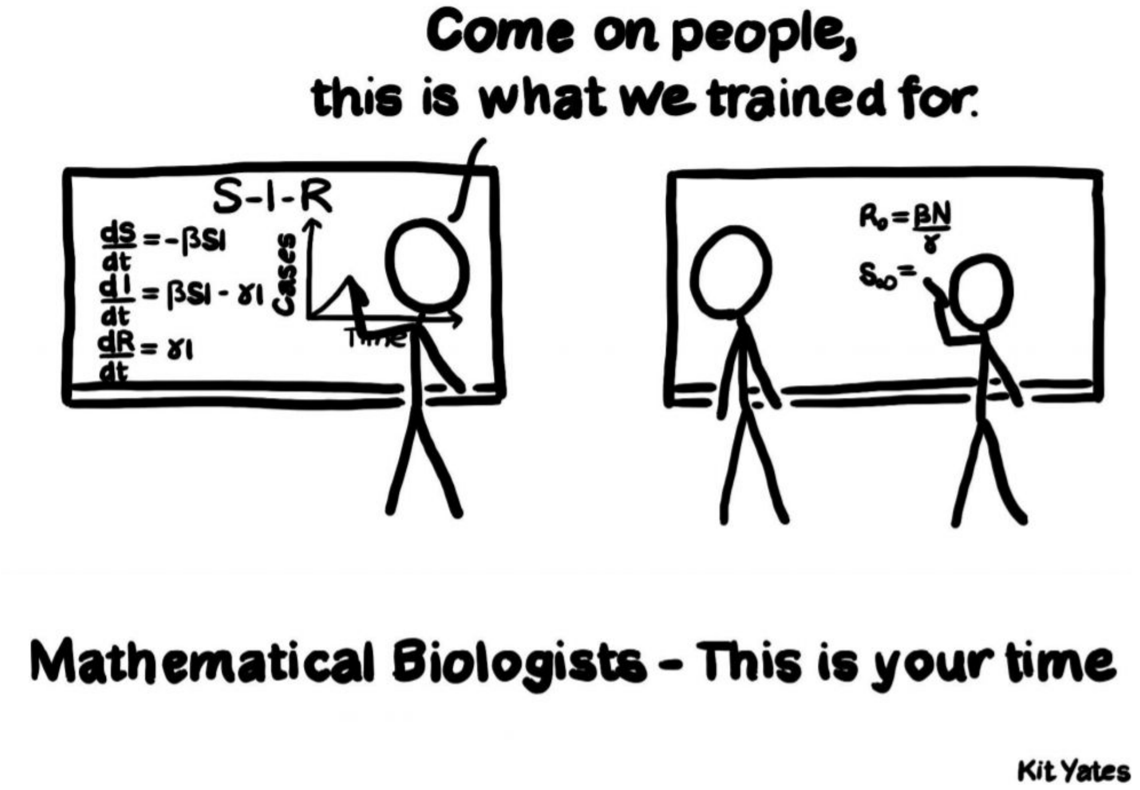
My Journey

Pre-uni interest: “pure” maths, maybe theoretical physics, curious about mathematical-biology

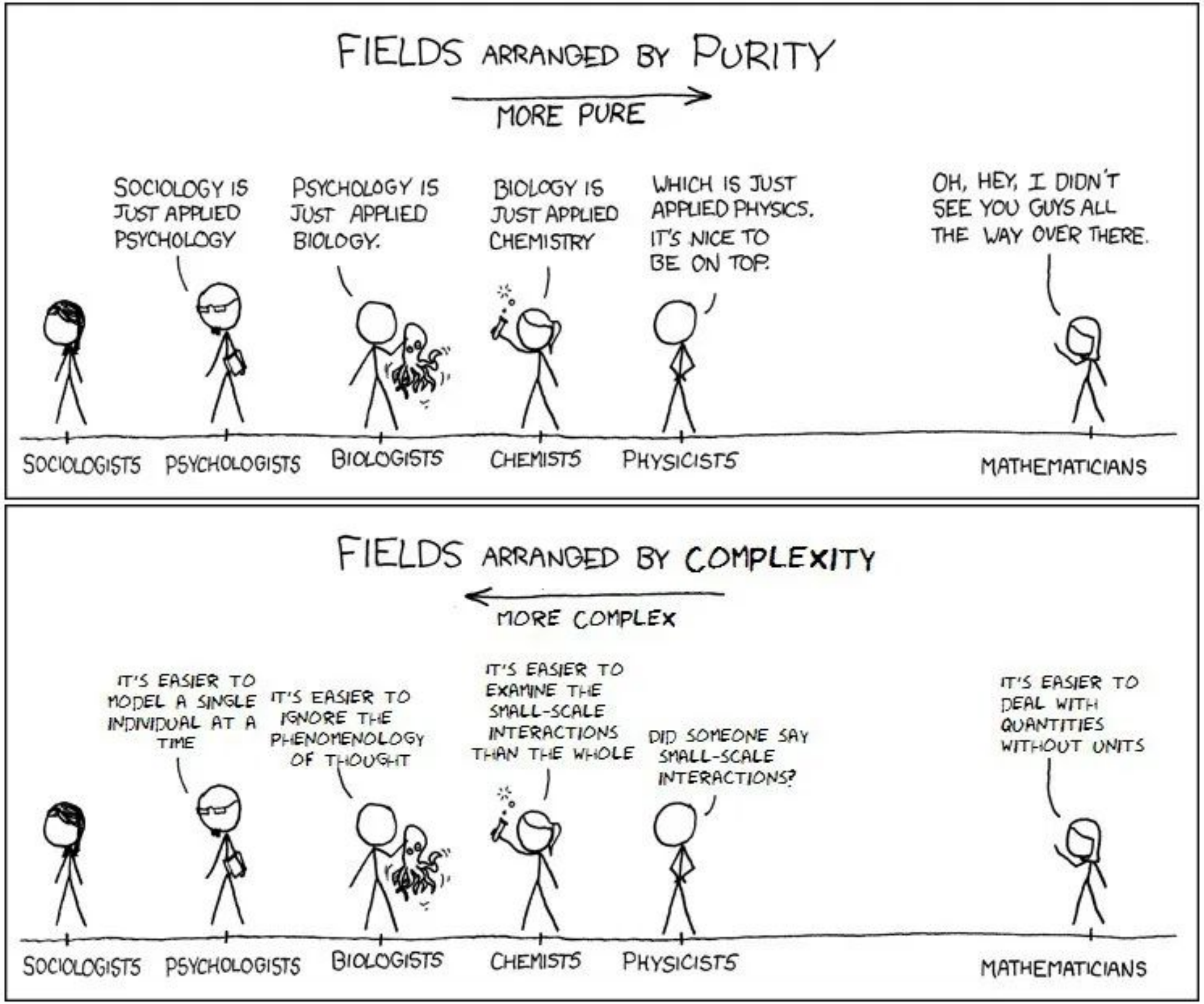
Applied maths → Optimization → Probability/applied analysis →
Summer **research** project in **mathematical biology** (population dynamics) →
Quantum Mechanics → General Relativity →
Summer **research** project in theoretical **cosmology** →
Part III: theoretical physics courses + “physics-epidemiology” essay?! →

Next: PhD at DAMTP in **mathematical epidemiology**

Why mathematical biology?



Now, seriously...



Mathematics Is Biology's Next Microscope, Only Better; Biology Is Mathematics' Next Physics, Only Better

Joel E. Cohen

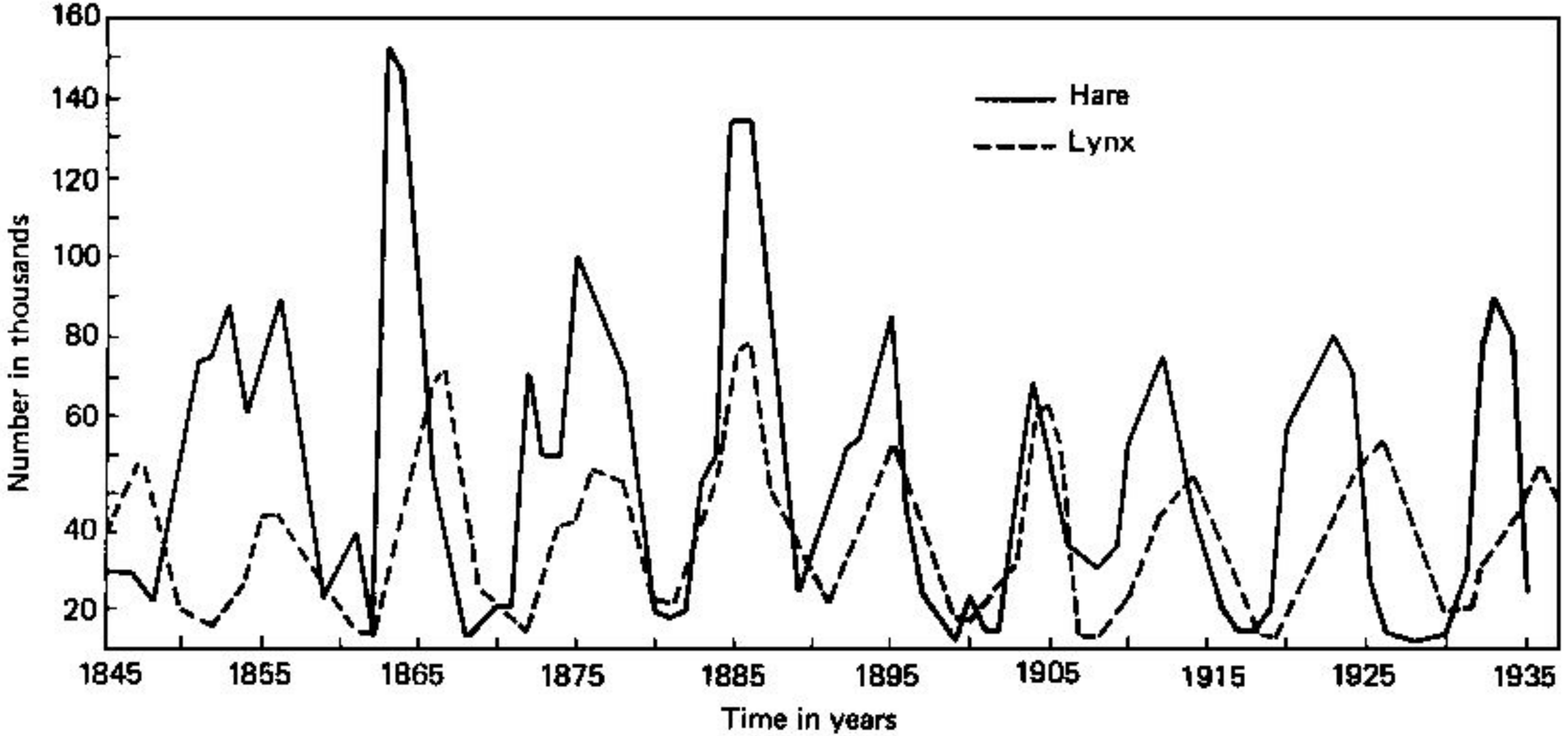
History...

Age structure of stable populations	Euler 1760
Logistic equation for limited population growth	Verhulst 1838
Branching processes, extinction of family names	Galton 1889
Correlation	Pearson 1903
Markov chains, statistics of language	Markov 1906
Hardy–Weinberg equilibrium in population genetics	Hardy 1908; Weinberg 1908
Analysis of variance, design of agricultural experiments	Fisher 1950
Dynamics of interacting species	Lotka 1925; Volterra 1931
Birth process, birth and death process	Yule 1925; Kendall 1948, 1949
Traveling waves in genetics	Fisher 1937; Kolmogorov et al. 1937
Game theory	von Neumann and Morgenstern 1953

What sort of maths?

- Applied maths
    - Systems of differential equations
    - Partial differential equations
    - …
- Dynamical systems and chaos theory
- Applied Probability
- Statistics
- …
- Biophysics / biomechanics / fluid dynamics...

Population Dynamics



Exponential growth: model formulation

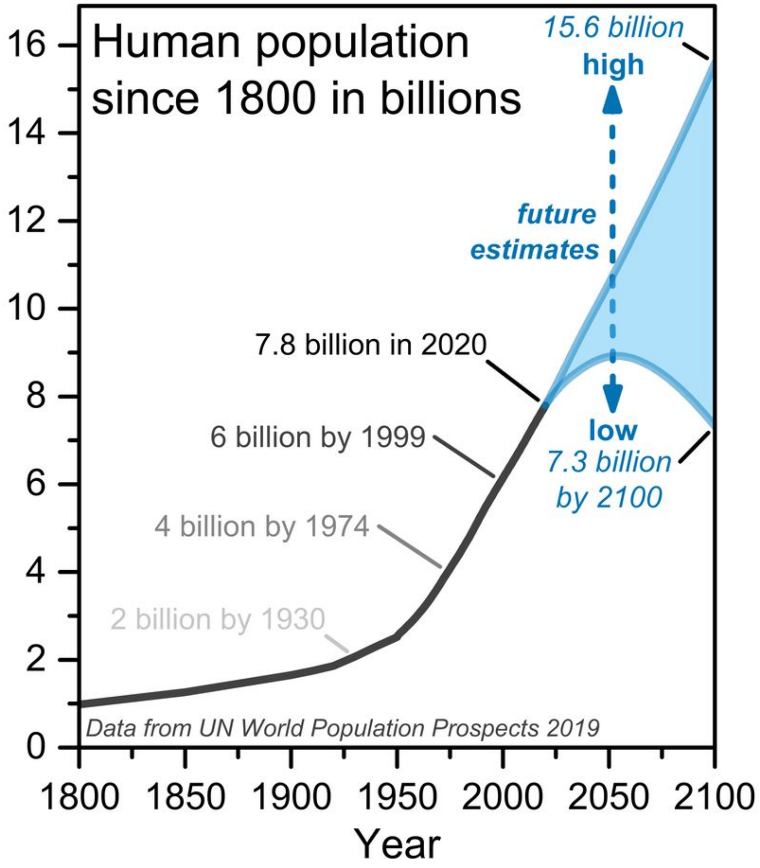
N_t population at time t

$$N_{t+1} = (1 + b - d)N_t$$

$$N_{t+1} = rN_t$$

$$r = 1 + b - d$$

b birth rate (per capita)
d death rate (per capita)



Exponential growth: solving the model $\boxed{N_{t+1} = rN_t}$

$$N_{t+1} = rN_t,\ N_t = rN_{t-1}$$

$$N_{t+1} = rrN_{t-1} = r^2 N_{t-1}$$

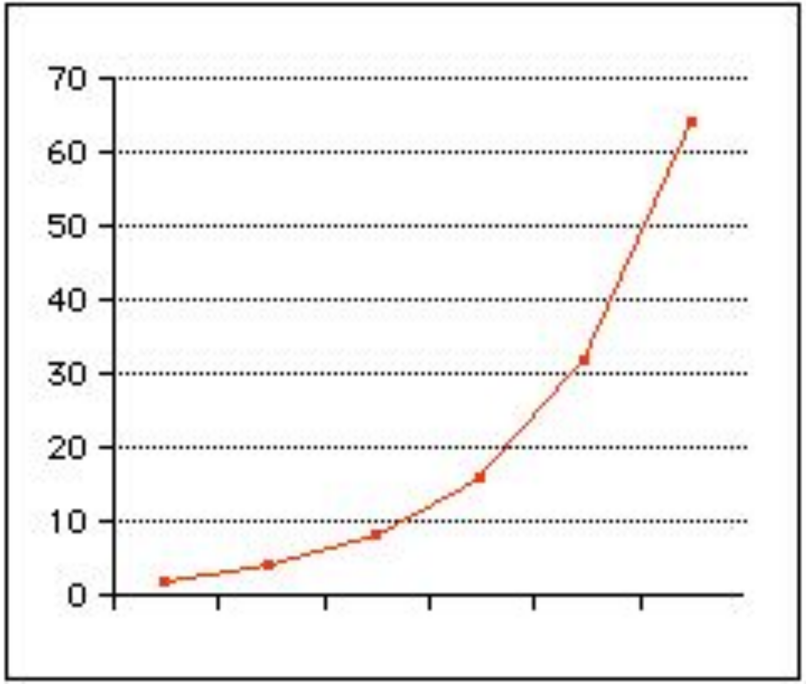
$$N_{t+1} = r^2 rN_{t-2} = r^3 N_{t-2}$$

$$N_{t+1} = r^{t+1} N_0$$

$$\boxed{N_t = r^t N_0}$$

t=0,1,2,3….

Geometric Growth = Discrete exponential growth

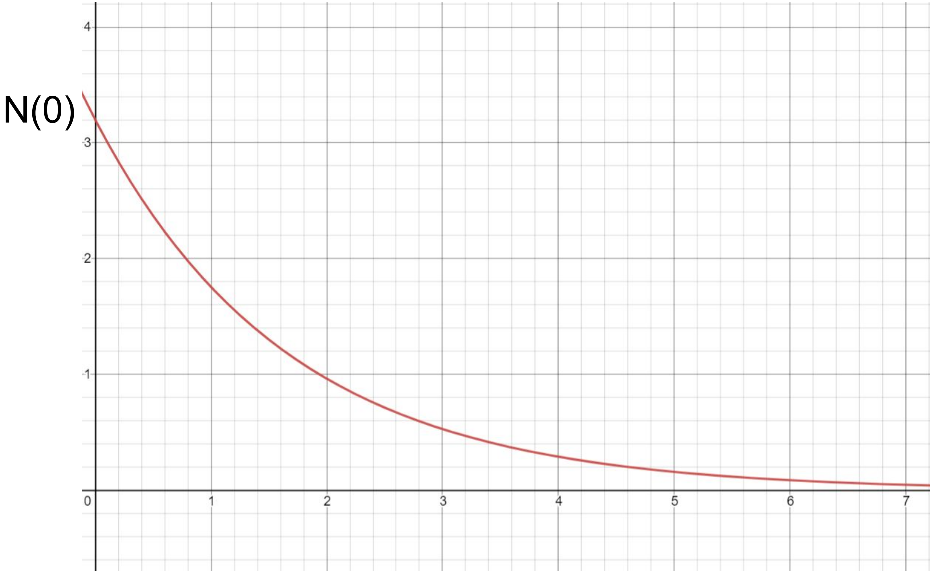


Continuous analogy

$$\frac{dN}{dt} = \dot{N}(t) = (b-d)N(t) => N(t) = N(0)e^{(b-d)t}$$

(b-d) is the **growth rate**

This is called **Malthusian growth**



Analysing the solution

$$N_t = r^t N_0$$

- If **r>1**

$$r^t > 1 => N_t > N_0$$

growth

- If **r<1**

$$r^t < 1 => N_t < N_0$$

decay

- If **r=1**

$$r^t = 1 => N_t = N_0$$

constant

Limiting behaviour

$$N_t = r^t N_0$$

- If **r>1**

$$N_t \to \infty \text{ as } t \to \infty$$

Infinite exponential growth

- If **r<1**

$$N_t \to 0 \text{ as } t \to \infty$$

Extinction

- If **r=1**

$$N_t = N_0 \forall t$$

Constant size in the population

Biological interpretation

We’ve seen that the evolution of the population depends on r

recall r=1-d+b

- So **r<1 if and only if** (iff) **b<d...more deaths than births...**eventual extinction
- Similarly, **r>1 iff b>d… more births than deaths…** growth
- Finally, if **birth rate = death rate…** population stays constant

So analytical results make sense with biological interpretation… do they?

Infinite growth?

If r>1, we’ve seen that the population grows exponentially (ie very fast)...
to infinity?

Can populations grow that much?

Only if they have infinity resources...this never happens in practice…

At some point, they have to start competing…

$$N_{t+1} = rN_t - cN_tN_t$$

Solution?

$$N_{t+1} = rN_t - cN_t^2$$

Can't do the same trick as before…

But can we say something?

Fixed points:

$$N_{t+1} = N_t => N_t = rN_t - cN_t^2$$

$$N_t = 0 \text{ or}$$

$$1 = r - cN_t => N_t = \frac{r-1}{c} = \frac{b-d}{c}$$

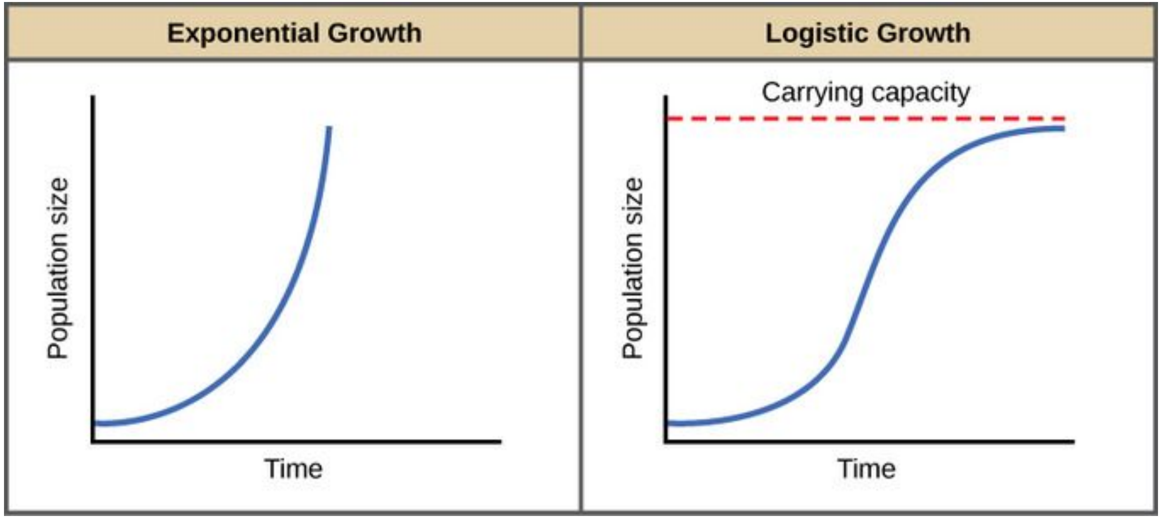
Logistic growth continuous version

s=b-d, K = “carrying capacity”

$$\frac{dN}{dt} = sN(1 - N/K)$$

$$N(t) = \frac{KN(0)e^{st}}{K - N(0) + N(0)e^{st}}$$

fixed points by setting the time derivative to 0: N=0 or N=K

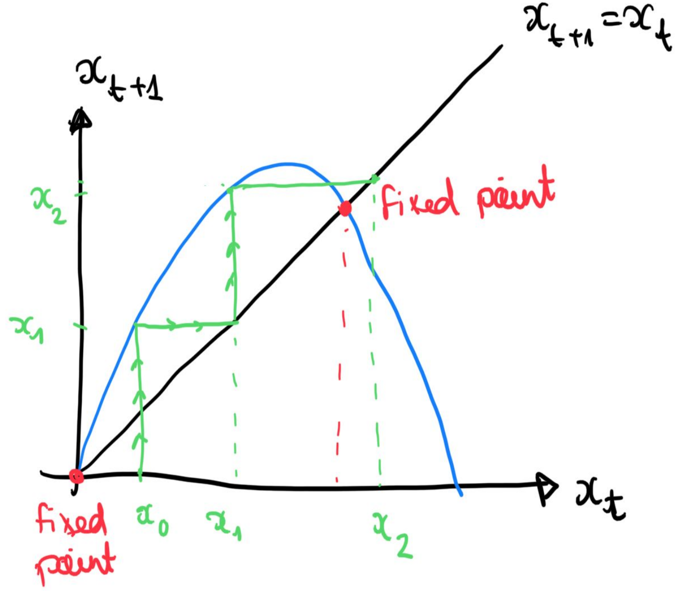


Can we study the trajectory?

$$N_{t+1} = rN_t - cN_t^2$$

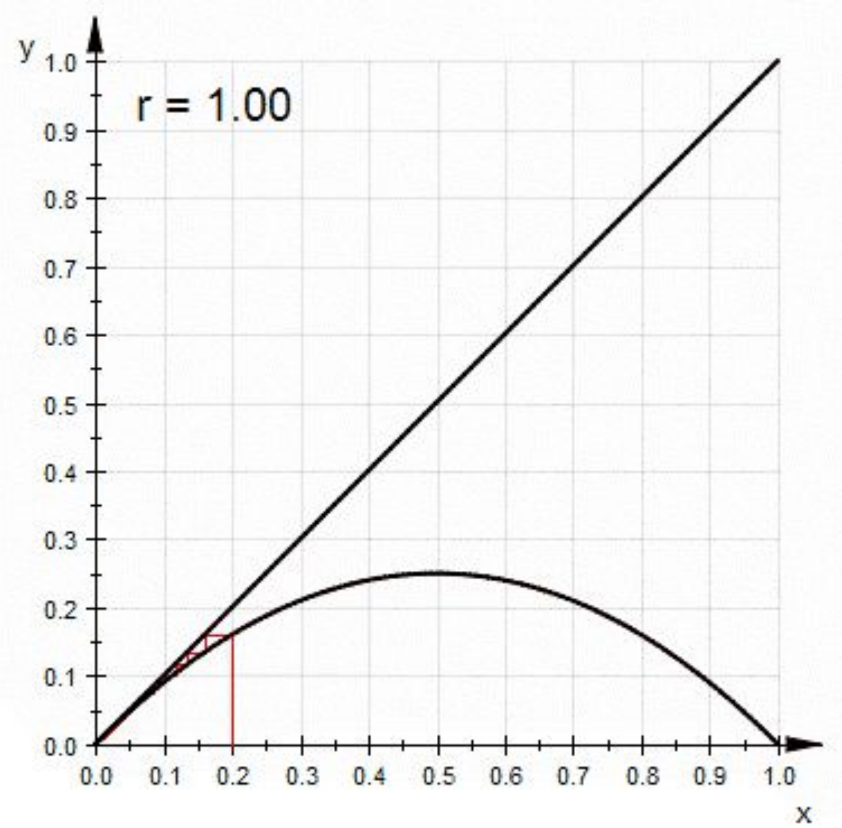
Rewrite as $x_{t+1} = rx_t(1 - x_t)$

$x_t = cN_t/r$



Iterations of the logistic map

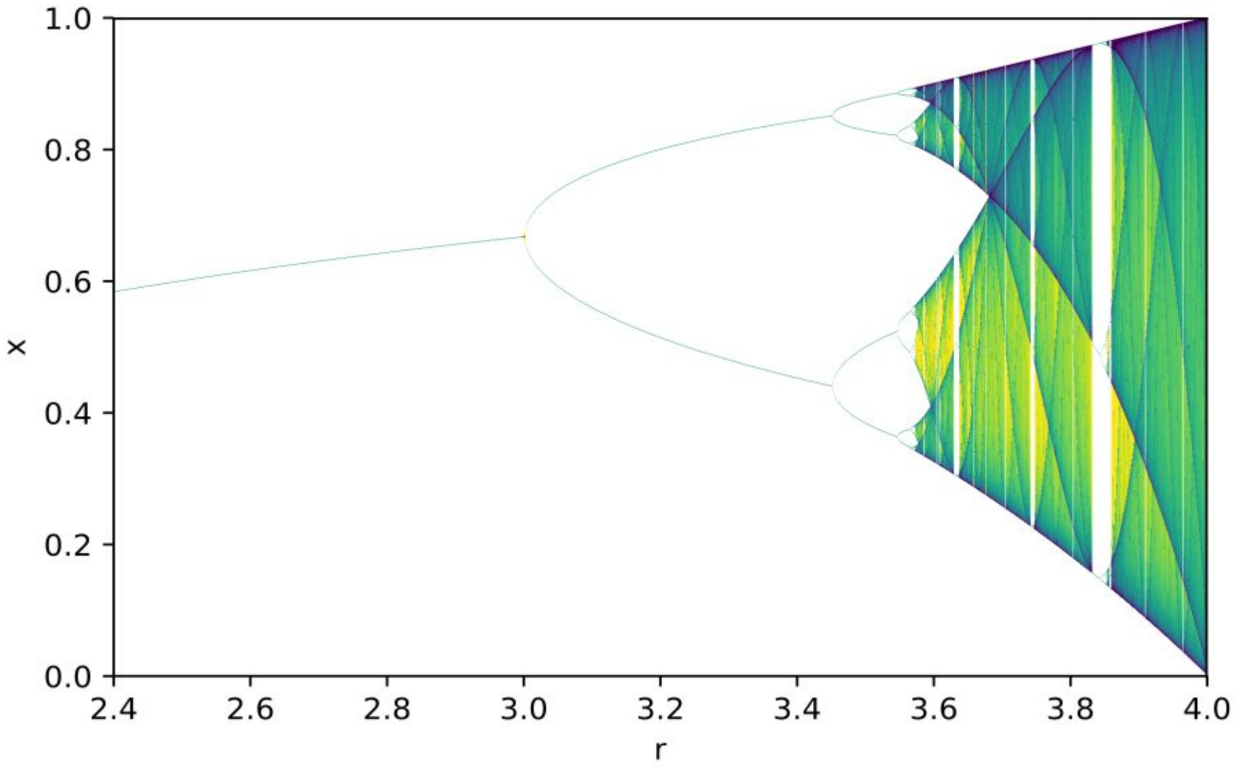
Chaos!



y
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
r = 1.00
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
x

Bifurcation diagram

Periodic orbits? $x_{t+k} = x_t$



Fibonacci's Rabbits

A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

$y_{k,n}$ Number of k-months old pairs of rabbits at time n

y_n Number of pairs of rabbits at time n

$$y_n = \sum_{k=0}^{\infty} y_{k,n}$$

$$\begin{aligned} y_{n+2} &= y_{1,n+2} + y_{2,n+2} + y_{3,n+2} + \cdots \\ &= (y_{2,n+1} + y_{3,n+1} + \cdots) + (y_{1,n+1} + y_{2,n+1} + \cdots) \\ &= (y_{1,n} + y_{2,n} + \cdots) + y_{n+1} = y_n + y_{n+1}, \end{aligned}$$

$$\text{i.e. } y_{n+2} = y_{n+1} + y_n$$

Solution

$$y_{n+2} = y_{n+1} + y_n$$

This is a **second order *difference* equation**

Need two initial conditions to fully specify the answer

Try a solution $y_n = \delta^n$ so $\delta^{n+2} = \delta^{n+1} + \delta^n =>$

$$\delta^2 = \delta + 1$$

$$\delta_\pm = \frac{1}{2}(1 \pm \sqrt{5})$$

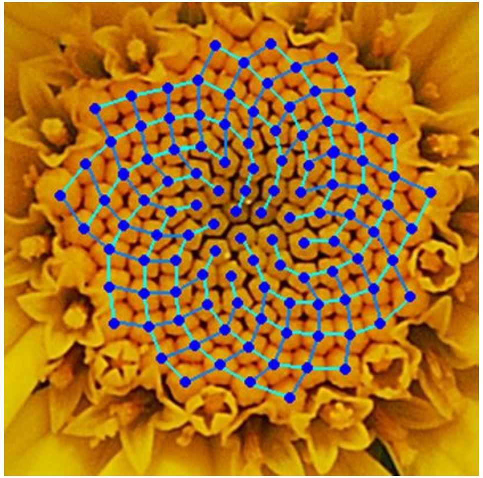
$$y_n = A\delta_+^n + B\delta_-^n$$

Initial conditions

Let us assume the initial pair is adult, two months old without loss of generality. The initial conditions are $y_{1,0} = 0$, $y_{2,0} = 1$, $y_{k,0} = 0$ for $k > 2$. So

Our single initial condition for the total population is $y_0 = 1$, which is not enough to solve this second order equation. However, it is easy to see that $y_{1,1} = 0$, $y_{2,1} = 0$, $y_{3,1} = 1$, $y_{k,1} = 0$ for $k > 3$, so that $y_1 = 1$. We have initial conditions

$$y_0 = y_1 = 1$$



The resulting sequence is 1,1,2,3,5,8,13,...

Fibonacci numbers (found in nature)

Golden ratio

$$\boxed{y_0 = y_1 = 1}$$

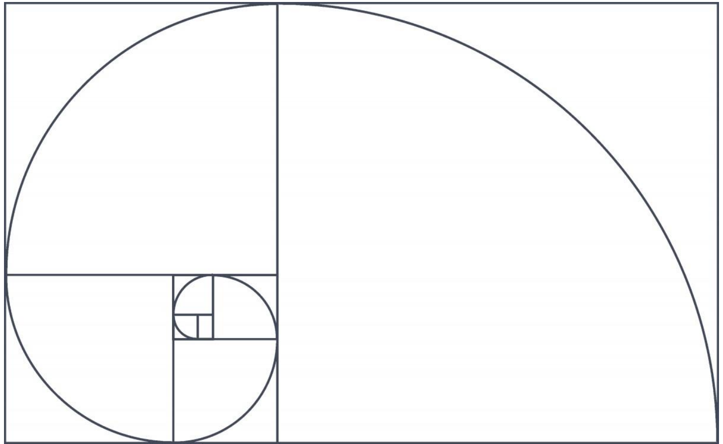
$$\boxed{y_n = A\delta_+^n + B\delta_-^n}$$

$$\delta_\pm = \frac{1}{2}(1 \pm \sqrt{5})$$

$$A = (1+\sqrt{5})/(2\sqrt{5}), B = -(1-\sqrt{5})/(2\sqrt{5})$$

$$y_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

$$y_n \approx A\delta_+^n \text{ for large n}$$



Any questions?

Thank you for listening <3

Feedback form

https://forms.gle/99miFMJJgKm2Ugk29